Kyungpook Math. J.
Volume 26, Number 1
June, 1986

# AN APPLICATION OF THE FRACTIONAL DERIVATIVE IV

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## 1. Introduction

Let $A$ denote the class of functions

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

analytic in the unit disk $U=\{|z|<1\}$. A function $f(z)\in A$ is said to be univalent and starlike if, and only if,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0$$

for $z\in U$. Recently, the general idea of order $k$ for a starlike function has been introduced in a number of ways as in M.S. Robertson [10], R.J. Libera [4], K.S. Padmanabhan [9] and F. Holland and D.K. Thomas [3]. In this place, according to K.S. Padmanabhan [9], a function $f(z)\in A$ is said to be starlike of order $k$ in the unit disk $U$ if the condition

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-1}{\dfrac{zf'(z)}{f(z)}+1}\right|<k$$

holds for some $k(0<k\leqq 1)$ and $z\in U$. The class of such functions we denote by $S(k)$. And let $C(k)$ denote the class of functions $f(z)\in A$ such that $zf'(z)\in S(k)$. For these classes $S(k)$ and $C(k)$, the author showed the following results in [7].

LEMMA 1. *A function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

*is in the class $S(k)$ if, and only if,*

$$\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}a_n \leqq 2k.$$

*The equality holds for the function*

$$f(z)=z-\sum_{n=2}^{\infty}\frac{2k}{(n-1)+k(n+1)}z^n$$

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AMS(MOS) subject classifications(1980) 26A24, 30C45.

LEMMA 2. *A function*

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geqq 0)$$

*is in the class $C(k)$ if and only if,*

$$\sum_{n=2}^{\infty}n\{(n-1)+k(n+1)\}a_n\leqq 2k.$$

*The equality holds for the function*

$$f(z)=z-\sum_{n=2}^{\infty}\frac{2k}{n\{(n-1)+k(n+1)\}}z^n.$$

LEMMA 3. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geqq 0)$$

*be in the class $S(k)$. Then we have*

$$|z|-\frac{2k}{1+3k}|z|^2\leqq|f(z)|\leqq|z|+\frac{2k}{1+3k}|z|^2$$

*and*

$$1-\frac{4k}{1+3k}|z|\leqq|f'(z)|\leqq 1+\frac{4k}{1+3k}|z|$$

*for $z\in U$. The equalities hold for the function*

$$f(z)=z-\frac{2k}{1+3k}z^2.$$

LEMMA 4. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geqq 0)$$

*be in the class $C(k)$. Then we have*

$$|D_z^{\alpha}f(z)|\geqq\frac{|z|^{1-\alpha}(1+3k-2k|z|)}{(1+3k)\Gamma(2-\alpha)}$$

*and*

$$|D_z^{\alpha}f(z)|\leqq\frac{|z|^{1-\alpha}(1+3k+2k|z|)}{(1+3k)\Gamma(2-\alpha)}$$

*for $0<\alpha<1$ and $z\in U$, where $D_z^{\alpha}f(z)$ denotes the fractional derivative of order $\alpha$ of $f(z)$.*

Furthermore K. S. Padmanabhan [9] gave the following representation formula.

LEMMA 5. *A function*

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geqq 0)$$

*is in the class* $S(k)$ *if and only if,*

$$f(z)=z\ \exp\left\{-2\int_0^z \frac{\phi(t)}{1+t\phi(t)}dt\right\},$$

*where* $\phi(z)$ *is analytic in the unit disk* $U$ *and satisfies* $|\phi(z)|\leqq k$ *for* $z\in U$.

## 2. An application of the fractional derivative

There are many definitions of the fractional calculus, that is, the fractional derivatives and the fractional integrals. The author gave the following definitions for the fractional derivative in [8].

DEFINITION 1. The fractional derivative of order $\alpha$ is defined by

$$D_z^\alpha f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^\alpha},$$

where $0<\alpha<1$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0} D_z^\alpha f(z).$$

DEFINITION 2. Under the hypotheses of Definition 1, the fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha}f(z)=\frac{d^n}{dz^n}D_z^\alpha f(z),$$

where $0<\alpha<1$ and $n\in N\cup\{0\}$.

For other definitions of the fractional calculus, see R. P. Agarwal [1], W. A. Al-Salam [2], T. J. Osler [6], B. Ross [11], K. Nishimoto [5] and M. Saigo [12].

THEOREM 1. *Let* $0<\alpha\leqq 1/2$ and $\alpha\leqq k\leqq 1$. *If a function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n\geqq 0)$$

*is in the class* $S(k)$, *then we have*

$$\frac{|z|^{1-\alpha}(1-2k|z|)}{\Gamma(2-\alpha)}\leqq|D_z^\alpha f(z)|\leqq\frac{|z|^{1-\alpha}(1+2k|z|)}{\Gamma(2-\alpha)}$$

for $z\in U$.

PROOF. We consider the function

$$G(z)=\Gamma(2-\alpha)z^\alpha D_z^\alpha f(z)$$
$$=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n z^n.$$

Then we obtain

$$|G(z)|\geqq|z|-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n|z|^n$$

$$\geqq|z|-|z|^2\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}a_n$$

for $0<\alpha\leqq 1/2$ and $\alpha\leqq k\leqq 1$. Hence, by Lemma 1, we get

$$|G(z)|\geqq|z|(1-2k|z|)$$

which gives

$$|D_z^{\alpha}f(z)|\geqq\frac{|z|^{1-\alpha}(1-2k|z|)}{\Gamma(2-\alpha)}.$$

In the same way, we have

$$|G(z)|\leqq|z|+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n z^n$$

$$\leqq|z|+|z|^2\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}a_n$$

$$\leqq|z|(1+2k|z|)$$

which gives

$$|D_z^{\alpha}f(z)|\leqq\frac{|z|^{1-\alpha}(1+2k|z|)}{\Gamma(2-\alpha)}.$$

COROLLARY 1. *Let* $0<\alpha\leqq 1/2$ *and* $\alpha\leqq k\leqq 1$. *If a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n\geqq 0)$$

*is in the class* $S(k)$, *then* $D_z^{\alpha}f(z)$ *is included in the disk with center at the origin and radius* $(1+2k)/\Gamma(2-\alpha)$.

THEOREM 2. *Let* $0<\alpha\leqq 1/2$ *and* $\alpha\leqq k\leqq 1$. *If a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n\geqq 0)$$

*is in the class* $C(k)$, *then we have*

$$|D_z^{1+\alpha}f(z)|\geqq\frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\left\{(1-\alpha)-\frac{2k(1+3k+\alpha)|z|}{1+3k}\right\}$$

*and*

$$|D_z^{1+\alpha}f(z)|\leqq\frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\left\{(1+\alpha)+\frac{2k(1+3k+\alpha)|z|}{1+3k}\right\}$$

*for* $z\in U-\{0\}$.

PROOF. We consider the function

$$G(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}f(z)$$

$$=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_nz^n.$$

Then, by using Lemma 2, we obtain

$$\begin{aligned}|G'(z)|&\geqq 1-\sum_{n=2}^{\infty}\frac{n\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n|z|^{n-1}\\&\geqq 1-|z|\sum_{n=2}^{\infty}n\{(n-1)+k(n+1)\}a_n\\&\geqq 1-2k|z|\end{aligned}$$

and

$$\begin{aligned}|G'(z)|&\leqq 1+\sum_{n=2}^{\infty}\frac{n\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n|z|^{n-1}\\&\leqq 1+|z|\sum_{n=2}^{\infty}n\{(n-1)+k(n+1)\}a_n\\&\leqq 1+2k|z|\end{aligned}$$

for $0<\alpha\leqq 1/2$ and $\alpha\leqq k\leqq 1$. Consequently, with the aid of Lemma 4, we get

$$\begin{aligned}|D_z^{1+\alpha}f(z)|&\geqq\frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}(1-2k|z|)-\alpha|z|^{-1}|D_z^{\alpha}f(z)|\\&\geqq\frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\left\{(1-\alpha)-\frac{2k(1+3k+\alpha)|z|}{1+3k}\right\}\end{aligned}$$

and

$$\begin{aligned}|D_z^{1+\alpha}f(z)|&\leqq\frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}(1+2k|z|)+\alpha|z|^{-1}|D_z^{\alpha}f(z)|\\&\leqq\frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\left\{(1+\alpha)+\frac{2k(1+3k+\alpha)|z|}{1+3k}\right\}\end{aligned}$$

for $z\in U-\{0\}$.

THEOREM 3. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geqq 0)$$

*be in the class $C(k)$. Then we have*

$$D_z^{\alpha}f(z)=\frac{z^{1-\alpha}}{\Gamma(2-\alpha)}exp\left\{-2\int_0^z\frac{\phi(t)}{1+t\phi(t)}dt\right\},$$

*where $\phi(z)$ is an analytic function in the unit disk $U$ and satisfies $|\phi(z)|\leqq k$ for $z\in U$.*

PROOF. We consider the function

$$\begin{aligned}G(z)&=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}f(z)\\&=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_nz^n\end{aligned}$$

$$=z-\sum_{n=2}^{\infty}A_n z^n$$

where

$$A_n=\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n$$

Then, by using Lemma 2, we have

$$\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}A_n \leqq \sum_{n=2}^{\infty}n\{(n-1+k(n+1)\}a_n$$
$$\leqq 2k,$$

becaues $0<A_n<na_n$ for any $n\geqq 2$. Hence the function $G(z)$ belongs to the class $S(k)$ with the aid of Lemma 1. Thus we obtain

$$D_z^{\alpha}f(z)=\frac{z^{1-\alpha}}{\Gamma(2-\alpha)}\exp\left\{-2\int_0^z \frac{\phi(t)}{1+t\phi(t)}dt\right\}$$

by using Lemma 5.

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## REFERENCES

[1] R.P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Cambridge Philos. Soc., 66 (1969), 365—370.

[2] W.A. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edinburgh Math. Soc., 15 (1966), 135—140.

[3] F. Holland and D.K. Thomas, *On the order of a starlike function*, Trans. Amer. Math. Soc., 158 (1971), 189—201.

[4] R.J. Libera, *Some radius of convexity problems*, Duke Math. J., 31 (1961), 143—158.

[5] K. Nishimoto, *Fractional derivative and integral Part* I, J. Coll. Engng. Nihon Univ., 17 (1976), 11—19.

[6] T.J. Osler, *Leibniz rule for fractional derivative generalized and an application to infinite series*, SIAM J. Appl. Math., 16 (1970), 658—674.

[7] S. Owa, *On certain classes of univalent functions in the unit disk* (preprint).

[8] S. Owa, *On the distortion theorems* I, Kyungpook Math. J., 18 (1978), 53—59.

[9] K.S. Padmanabhan, *On certain classes of starlike functions in the unit disc*, J. Indian Math. Soc., 32 (1968), 89—103.

[10] M.S. Robertson, *On the theory of univalent functions*, Ann. Math., 37 (1936), 374—408.

[11] B. Ross, *A brief history and exposition of the fundamental theory of fractional calculus*, Lecture Notes in Mathematics, 457 (1975), 1—36.

[12] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ., 11 (1978), 135—143.