Kyungpook Math. J.
Volume 26, Number 1
June, 1986

ON DISTRIBUTIVE AND MODULAR NEARLATTICES

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1. Introduction and basic concepts

A meetsemilattice S, where any two elements b and c have a least upper bound $b\vee c$ whenever there is an upper bound for b and c in S is called in [3] a nearlattice. As reported in [3], nearlattices constitute a natural generalization of lattices, and their ideals provide a handy tool for analyzing nearlattices. The purpose of this paper is to describe the structure of nearlattices by means of appropriate algebras, ideals, dual ideals and multiplicative closure operators.

At first we consider an algebra associated with a nearlattice N. Because N is a meetsemilattice, we define $b\wedge c=\mathrm{glb}\{b,c\}$, the greatest lower bound of b and c, for any two $b,c\in N$. Further, if there is an element $d\geq b,c$ in N, we define $b\vee c=\mathrm{lub}\{b,c\}$, the least upper bound of b and c. If the set $\mathrm{ub}\{b,c\}$ is empty, we define $b\vee c=b\wedge c$. Thus a nearlattice N is an algebra with two binary operations $\vee$ and $\wedge$. The properties of the operations characterize a nearlattice completely as stated in the following two lemmas.

LEMMA 1. *Let the poset* $(P,\geq)$ *be a nearlattice. Then the nearlattice* $(P,\vee,\wedge)=N$ *is an algebra with operations* $\wedge$ *and* $\vee$, *where* $a\wedge b=glb\{a,b\}$ *and* $a\vee b=lub\{a,b\}$ *whenever* $ub\{a,b\}\neq\phi$ *and otherwise* $a\vee b=a\wedge b$, *satisfying the following properties for all* $a,b,c\in P$:

(1) $a\wedge a=a$; (1′) $a\vee a=a$;

(2) $a\wedge b=b\wedge a$; (2′) $a\vee b=b\vee a$;

(3) $a\wedge(b\wedge c)=(a\wedge b)\wedge c$;

(4) *if* $a\neq b$, $c\wedge a=a$ *and* $c\wedge b=b$, *then* $a\wedge b\neq a\vee b$;

(5) *if* $a\vee c=b\vee c=c$, *then* $(a\vee b)\vee c=a\vee(b\vee c)$;

(6) *if* $a\vee b=c$, *or if* $a\vee b\neq c$ *and* $c\vee(a\vee b)\neq c\wedge(a\vee b)$, *then* $(a\wedge b)\wedge((a\vee b)\vee c)=a\wedge b$;

(7) $a\wedge b=a\Leftrightarrow a\vee b=b$.

PROOF. (1), (2) and (3) follow from the fact that $(P,\geq)$ is a meetsemilattice. (1′) and (2′) are trivial. In (4) $c\wedge a=a$ and $c\wedge b=b$ imply that $c\in\mathrm{ub}\{a,b\}$ in $(P,\geq)$, whence $a\vee b=\mathrm{lub}\{a,b\}$. Now, when $a\neq b$, $\mathrm{glb}\{a,b\}\cap\mathrm{lub}\{a,b\}=\phi$, and thus $a\wedge b\neq a\vee b$. If $a\vee b=c$ in (6), then obviously $(a\wedge b)\wedge((a\vee b)\wedge c)=a\wedge b$. If $a\vee b\neq c$ and $c\vee(a\vee b)\neq c\wedge(a\vee b)$, then $\mathrm{ub}\{a\vee b,c\}=\phi$ and thus $((a\vee b)\vee c)\geq a\vee b$ in $(P,\geq)$. This implies $(a\wedge b)\wedge((a\vee b)\vee c)=a\wedge b$, and (6) follows.

We show next the validity of (7). If $a\wedge b=a$, then $a\leq b$ in $(P,\geq)$, and thus $a\vee b=\mathrm{lub}\{a,b\}=b$. This proves $\Rightarrow$; the proof for $\Leftarrow$ is similar when $\mathrm{lub}\{a,b\}\neq\phi$. So let $\mathrm{lub}\{a,b\}=\phi$, and thus $b=a\vee b=a\wedge b$, whence $b\in\mathrm{glb}\{a,b\}$ and $b\leq a$ in $(P,\geq)$. But then $a\in\mathrm{lub}\{a,b\}\neq\phi$, which is a contradiction. Hence (7) holds. At last we prove (5). When $a\vee c=b\vee c=c$, then $a,b\leq c$, and thus $a\vee b=\mathrm{lub}\{a,b\}\leq c$. Then, in particular, $(a\vee b)\vee c=c=a\vee c=a\vee(b\vee c)$.

LEMMA 2. *If $(P,\vee,\wedge)$ is an algebra with binary operations $\vee$ and $\wedge$ satisfying* (1)~(7), (1′) *and* (2′), *i.e. a nearlattice, then $a\wedge b=a\Leftrightarrow a\leq b$ determines a partial order on P and the poset $(P,\geq)$ is a nearlattice. If the nearlattice $(P,\geq)$ determines a nearlattice $(P,\vee,\wedge)=N$, and if N determines further a nearlattice $(P,\gtrsim)$, then $(P,\geq)=(P,\gtrsim)$. Moreover, if $(P,\gtrsim)$ determines a nearlattice $(P,\cup,\cap)=N'$, then $N=N'$.*

PROOF. According to (1)~(3) the relation $\geq$ given by the rule "$a\leq b\Leftrightarrow a\wedge b=a$" is a partial order on P, and $a\wedge b=\mathrm{glb}\{a,b\}$ in the poset $(P,\geq)$ derived from $(P,\vee,\wedge)$. So it remains to show that $a\vee b=\mathrm{lub}\{a,b\}$ if $\mathrm{ub}\{a,b\}\neq\phi$. If $a=b$, then $a\vee a=a=\mathrm{lub}\{a,b\}$. Hence we assume that $a\neq b$ and $a\vee b\neq a\wedge b$; (4) contradicts the assumption that $a\vee b=a\wedge b$ and $\mathrm{ub}\{a,b\}\neq\phi$ for $a\neq b$. By (6) we obtain now that $a\wedge(a\vee b)=(a\wedge a)\wedge((a\vee a)\vee b)=a\wedge a=a$, whence $a\leq a\vee b$. Similarly we see that $b\leq a\vee b$. Thus $a\vee b\in\mathrm{ub}\{a,b\}\neq\phi$. If now there is an element $c\geq a,b$, then by (5) we see that $(a\vee b)\vee c=a\vee(b\vee c)=a\vee c=c$, whence $a\vee b=\mathrm{lub}\{a,b\}$. The assertion $(P,\geq)=(P,\gtrsim)$ follows now from the equivalences $a\leq b\Leftrightarrow a\wedge b=a\Leftrightarrow a\gtrsim' b$; also the assertion $N=N'$ is now obvionsly true.

In what follows, we write $N=(N,\vee,\wedge)$ for a nearlattice. Note that $\vee$ is not in general associative in a nearlattice.

2. Distributive and modular nearlattices.

A nearlattice $N=(N,\vee,\wedge)$ is distributive (modular) if and only if D_1 and D_2 (M_1 and M_2) below hold for all $a,b,c\in N$:

$D_1: a\wedge(b\vee c)\leq(a\wedge b)\vee(a\wedge c)$; $\quad D_2: a\vee(b\wedge c)\geq(a\vee b)\wedge(a\vee c)$;

$M_1: a\wedge(b\vee(a\wedge c))\leq(a\wedge b)\vee(a\wedge c)$; $\quad M_2: a\vee(b\wedge(a\vee c))\geq(a\vee b)\wedge(a\vee c)$.

Clearly every distributive N is also modular and every distributive (modular) lattice is also a distributive (modular) nearlattice. Note that every distributive meetsemilattice need not be a distributive nearlattice. The equality sign need not hold in D_1, D_2, M_1 and M_2; this can be seen by considering e.g. finite trees which are distributive nearlattices.

LEMMA 3. *Let N be a nearlattice. Then D_1 is equivalent to D_2 as well as M_1 to M_2.*

PROOF. We prove the implication $D_1 \Rightarrow D_2$ only; the other proofs are analogous and hence omitted. Let us consider the expression $(a \vee b) \wedge (a \vee c)$ in N. Three cases arise: 1) $a \vee b \neq \mathrm{lub}\{a, b\}$ and $a \vee c \neq \mathrm{lub}\{a, c\}$; 2) $a \vee b = \mathrm{lub}\{a, b\}$ and $a \vee c \neq \mathrm{lub}\{a, c\}$ (or $a \vee b \neq \mathrm{lub}\{a, b\}$ and $a \vee c = \mathrm{lub}\{a, c\}$); 3) $a \vee b = \mathrm{lub}\{a, b\}$ and $a \vee c = \mathrm{lub}\{a, c\}$.

1) $(a \vee b) \wedge (a \vee c) = a \wedge b \wedge a \wedge c = a \wedge b \wedge c \leq a \vee (b \wedge c)$, because $x \wedge y \leq x \vee y$ holds for all pairs $x, y \in N$.

2) Assume that $a \vee b = \mathrm{lub}\{a, b\}$ and $a \vee c \neq \mathrm{lub}\{a, c\}$. Because N is a meetsemilattice, $b \wedge c \leq b$, and because $a \vee b$ is now an upper bound of a and b, an upper bound of a and $b \wedge c$ also exists, whence $a \vee (b \wedge c) \geq a$. Now $a \vee (b \wedge c) \geq a \geq a \wedge c = (a \vee b) \wedge (a \wedge c) = (a \vee b) \wedge (a \vee c)$.

3)
$$\begin{aligned}(a \vee b) \wedge (a \vee c) &\leq ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) = a \vee ((a \wedge b) \vee c) \\ &\leq a \vee ((a \wedge c) \vee (c \wedge b)) = (a \vee (a \wedge c)) \vee (b \wedge c) \\ &= a \vee (c \wedge b).\end{aligned}$$

As expected, distributive (modular) nearlattices have the following structure

THEOREM 4. *A nearlattice N is distributive (modular) if and only if the set $(d] = \{x \mid x \in N$ and $x \leq d\}$ is a distributive (modular) lattice for every $d \in N$.*

PROOF. Assume that $(d]$ is a distributive lattice for all $d \in N$. when $a, b, c \in (d]$, then D_1 and D_2 hold for them. Assume now that a, b and c are three elements such that there is no $(d]$ in N containing them. When proving the validity of D_1 for a, b and c two cases arise: 1) $b \vee c \neq b \wedge c$ and 2) $b \vee c = b \wedge c$. When 2) holds we obtain: $a \wedge (b \vee c) = a \wedge (b \wedge c) = (a \wedge b) \wedge (a \wedge c) \leq (a \wedge b) \vee (a \wedge c)$. When 1) holds we write $t = a \wedge (b \vee c)$. Clearly $t, b, c, a \wedge c \in (b \vee c]$. In the lattice $(b \wedge c]$ we obtain now: $a \wedge (b \vee c) = t \wedge (b \vee c) = (t \wedge b) \vee (t \wedge c) = (a \wedge b) \vee (a \wedge c)$. Thus D_1 holds and by Lemma3 N is a distributive nearlattice. Conversely, when N is distributive, then the validity of D_1 in a lattice $(d]$ implies the distributivity of $(d]$, and the theorem follows in the distributive case.

The converse proof in the modular case is also obvious. When proving the first part in the modular case, two subcases arise: 1) $b\vee(a\wedge c)\neq b\wedge(a\wedge c)$ and 2) $b\wedge(a\wedge c)=b\vee(a\wedge c)$. The subcase 2) is obvious. In 1) we write $t=a\wedge(b\vee(a\wedge c))$ and observe that $t, a\wedge b, a\wedge c\ b\in(b\vee(a\wedge c)]$. Moreover, $t\wedge(a\wedge c)=a\wedge c$, and by using now the modularity in the lattice $(b\vee(a\wedge c)]$ we obtain $a\wedge(b\vee(a\wedge c))=(a\wedge b)\vee(a\wedge c)$. Hence M_1 holds and the modularity of N follows from Lemma 3. This completes the proof.

We call a nonempty set $I\subset N$ an ideal if and only if (i) and (ii) below hold: (i) $x\leq a\in I$ implies $x\in I$; (ii) $a, b\in I$ implies $a\vee b\in I$. One can easily see that the intersection of two ideals I and J of N is also an ideal and it is the greatest ideal contained in I and J. Hence the ideals of N constitute a lattice $I(N)$. It is known, that N is distributive if and only if $I(N)$ is a distributive lattice. It is also known that if $I(N)$ is modular, then N is modular, but the converse need not hold. Thus the modular case is of interest and in order to give a characterization we recall the concept of a relative annihilator. The annihilator $\langle a, b\rangle$ of a relative to b is the set $\{x\mid x\in N \text{ and } a\wedge x\leq b\}$ [2]. Mandelker proved in [2] that a lattice L is distributive if and only if $\langle a, b\rangle$ is an ideal for all $a, b\in L$. He proved further that L is modular if and only if whenever $b\leq a$, if $x\in(b]$ and $y\in\langle a, b\rangle$, then $x\vee y\in\langle a, b\rangle$. Both theorems can be proved for nearlattices but we recall the proof only in the modular case.

THEOREM 5. *A nearlattice N is modular if and only if whenever $b\leq a$, if $x\in(b]$ and $y\in\langle a, b\rangle$, then $x\vee y\in\langle a, b\rangle$.*

PROOF. Let N be modular, $b\leq a$, $x\in(b]$ and $y\in\langle a, b\rangle$. Then $x=x\wedge a\leq b$, $y\wedge a\leq b$, and thus $a\wedge(y\vee x)\leq(a\wedge x)\vee(a\wedge y)\leq b$, whence $x\vee y\in\langle a, b\rangle$. Conversely, let $x, y, z\in N$. Thus $x\wedge z\in((z\wedge x)\vee(z\wedge y)]$ and $y\in\langle z, (z\wedge x)\vee(z\wedge y)\rangle$. By hypothesis $(z\wedge x)\vee y\in\langle z, (z\wedge x)\vee(z\wedge y)\rangle$, i.e. $z\wedge((z\wedge x)\vee y)\leq(z\wedge x)\vee(z\wedge y)$. By Lemma 3 this proves the modularity of N.

The distributivity as well as the modularity of a nearlattice N can be characterized by means of their dual ideals. A nonempty set $D\subset N$ is called a dual ideal of N if (i) and (ii) below hold: (i) $x\geq d\in D$ implies $x\in D$; (ii) $a, b\in D$ implies $a\wedge b\in D$. A joinsemilattice, where any two elements having a lower bound have also a glb$\{a, b\}$ is called a dual nearlattice. The dual ideals $D(N)$ of a nearlattice N constitute obviously a dual nearlattice, where $D\vee J=\bigcap\{K\mid K\in D(N) \text{ and } D, J\subset K\}$. Moreover $D\wedge J=D\cap J$ if $D\cap J\neq\phi$, and otherwise $D\wedge J=D\vee J$ in $D(N)$. A dual nearlattice dN is called distributive (modular) if and only if D_1 and D_2 (M_1 and M_2) hold for all $a, b, c\in dN$. Obviously, Lemma 3 as well as the dual of Theorem 4 hold for dual nearlattices. Now we can prove

THEOREM 6. *A nearlattice N is distributive (modular) if and only if the dual nearlattice $D(N)$ of all dual ideals of N is distributive (modular).*

PROOF. As one can easily see, $D\vee J=\{x\mid x\geq d\wedge j,\ d\in D,\ j\in J\}$ for all $D, J\in D(N)$.

Assume first that N is distributive. We prove the distributivity of $D(N)$ by showing the distributivity of every lattice $[D)$ in $D(N)$. Let $J, M, J\in[D)$. Obviously $(M\vee J)\cap(M\vee I)\neq\phi$ and hence we may assume that $t\in(M\vee J)\wedge(M\vee I)$. Thus $t\geq m_1\wedge j'$ and $t\geq m_2\wedge i'$ with $m_1, m_2\in M, J'\in J$ and $i'\in I$. Clearly $t\geq m\wedge j,\ m\wedge i$, where $m=m_1\wedge m_2\in M$, and j and i can be choosen such that $j, i\leq d$ for some $d\in D$. Because of the distributivity of $N, t\geq(m\wedge j)\vee(m\wedge i)\geq m\wedge(j\vee i)$, where $m\in M$ and $j\vee i=\text{lub}\{i,j\}\in I\wedge J\supset D\neq\phi$. Thus $t\in M\vee(J\wedge I)$ and $(M\vee J)\wedge(M\vee I)\leq M\vee(J\wedge I)$, which proves the distributivity of $[D)$.

Let conversely $D(N)$ be a distributive dual nearlattice. We show the distributivity of N by showing the distributivity of the lattice $(d]\subset N$ for every $d\in N$. Let $a, b, c\in(d]$, whence $[a), [b), [c)\in[[d))$ in $D(N)$. The least element of $([a)\vee[b))\wedge([a)\vee[c))$ is $(a\wedge b)\vee(a\wedge c)$ and that of $[a)\vee([b)\wedge[c))$ is $a\wedge(b\vee c)$. Because $([a)\vee[b))\wedge([a)\vee[c))\leq[a)\vee([b)\wedge[c))$ in $D(N)$, we obtain $a\wedge(b\vee c)\leq(a\wedge b)\vee(a\wedge c)$, which implies the distributivity of $(d]$ in N.

The proof for the modular case is similar and hence omitted.

A convex subnearlattice S of a nearlattice N is a nonempty set such that if $a, b\in S$ then $a\vee b,\ a\wedge b\in S$, and if further $a\leq x\leq b$ then $x\in S$. Clearly $S=(S]\cap[S)$ and every convex subnearlattice S of N has a unique representation as the intersection of an ideal and a dual ideal of N. The convex subnearlattices S of a nearlattice N constitute a dual nearlattice $\text{Csub}(N)$, where $S_1\wedge S_2=S_1\cap S_2$ if $S_1\cap S_2\neq\phi$.

THEOREM 7. *A nearlattice N is distributive if and only if the dual nearlattice* $\text{Csub}(N)$ *is distributive. If $I(N)$ is a modular lattice, then* $\text{Csub}(N)$ *is a modular dual nearlattice, and if* $\text{Csub}(N)$ *is modular, then also N is modular.*

PROOF. Assume first that N is distributive; we show that $[S)$ is distributive for all $S\in\text{Csub}(N)$ from which the distributivity of $\text{Csub}(N)$ follows. Let $U, T, V\in[S)$. Now $(T\vee U)\wedge(T\vee V)=((T\vee U)\wedge(T\vee V)]\cap[(T\vee U)\wedge(T\vee V))=(T\vee U]\cap(T\vee U]\cap[T\vee U)\cap[T\vee V)$, where $(T\vee U]\cap(T\vee V]=((T]\vee(U])\cap((T]\vee(V])\leq(T]\vee((U]\vee(V])=(T]\vee(U\wedge V]=(T\vee(U\wedge V)]$ by the distributivity of $I(N)$, and analogously, $[T\vee U)\cap[T\vee V)\leq[T\vee(U\wedge V))$ by the distributivity of $D(N)$. Thus $(U\vee T)\wedge(T\vee V)\leq(T\vee(U\wedge V)]\cap[T\vee(U\wedge V))=T\vee(U\wedge V)$, which proves the distributivity of [S). The modularity of Csub(N) can be proved similarly.

Conversely, let Csub(N) be distributive. Then all dual ideals of N satisfy the distributivity laws and the distributivity of N can be proved as in the proof of Theorem 6. This holds also for the modular case.

Because a lattice is modular if and only if $I(L)$ is modular, we can write a corollary

COROLLARY 8. *A lattice is distributive (modular) if and only if the dual nearlattice* Csub (L) *of convex sublattices of* L *is distributive (modular).*

3. Closure operations

In this section we characterize the distributivity and modularity of nearlattices N by means of closure operators on N. Following Cornish [1] we call an operator λ on N a multiplicative closure operator if the conditions (i)~(iv) below hold for all $a,b\in N$: (1) $\lambda(a)\leq a$; (ii) $\lambda(\lambda(a))=\lambda(a)$; (iii) $a\leq b$ implies $\lambda(a)\leq\lambda(b)$; (iv) $\lambda(a\wedge b)=\lambda(a)\wedge\lambda(b)$. An operator π on N is called a translation on N if $\pi(a\wedge b)=\pi(a)\wedge b$ for all $a,b\in N$. As proved by Szász [4] every translation is a multiplicative closure operator on N. At first we characterize trees; a nearlattice N is a tree if and only if every two noncomparable elements have a common upperbound in N.

THEOREM 9. *A nearlattice N is a tree if and only if every multipltcative closure operator λ on N has the property $\lambda(a\vee b)\leq\lambda(a)\vee\lambda(b)$ for all $a,b\in N$.*

PROOF. Let N be a tree and λ a multiplicative closure operator on N. If $a\vee b=a\wedge b$, then $\lambda(a\vee b)=\lambda(a\wedge b)=\lambda(a)\wedge\lambda(b)\leq\lambda(a)\vee\lambda(b)$. If $a\vee b\neq a\wedge b$, then either $a\leq b$ or $b\leq a$. When $a\leq b$, we obtain $\lambda(a)\leq\lambda(b)$, and thus $\lambda(a\vee b)=\lambda(b)=\lambda(a)\vee\lambda(b)$. The proof is similar for $b\leq a$. Accordingly, $\lambda(a\vee b)\leq\lambda(a)\vee\lambda(b)$ for all $a,b\in N$.

Conversely, let the condition hold for every multiplicative closure operator on N, and assume that there exist two noncomparable elements $a.b\in N$ such that $a\wedge b\leq a,b\leq a\vee b$. We define a mapping $\lambda: N\to N$ as follows: if $t\geq a\vee b$ in N, then $\lambda(t)=a\vee b$, and otherwise $\lambda(\mathrm{t})=t\wedge a\wedge b$. By a direct computation one sees that λ is a multiplicative closure operator on N. But now $\lambda(a)\vee\lambda(b)=(a\wedge b)\vee(a\wedge b)=a\wedge b\ngeq a\vee b=\lambda(a\vee b)$, which is a contradiction. Hence two noncomparable elements cannot have a common upper bound in N, and thus N is a tree.

The following two theorems are modifications of corresponding results in [4]. Obviously a mapping $\pi_a(b)=a\wedge b$ is a translation for every $a\in N$ on a nearlattice N.

THEOREM 10. *A nearlattice N is distributive if and only if every translation π on N has the property $\pi(a\vee b)\leq\pi(a)\vee\pi(b)$ for all $a, b\in N$.*

The proof of Theorem 10 is analogous to that of Theorem 11 below, and hence we omit the proof of Theorem 10.

THEOREM 11. *A nearlattice N is modular if and only if every translation π on N has the property $\pi(a\vee b)\leq\pi(a)\vee\pi(b)$ for all $a, b\in N$ with $\pi(b)=b$.*

PROOF. Let N be modular and π a translation on N. If $a\vee b=a\wedge b$, then obviously $\pi(a\vee b)\leq\pi(a)\vee\pi(b)$. Thus let $a\vee b\neq a\wedge b$ and $b=\pi(b)$. Now $a\vee b\geq b$, whence $\pi(a\vee b)\geq\pi(b)=b$ as well as $\pi(a\vee b)\wedge\pi(b)=\pi(b)$. Then $\pi(a\vee b)=\pi((a\vee b)\wedge(a\vee b))=\pi(a\vee b)\wedge(a\vee b)=\pi(a\vee b)\wedge(a\vee\pi(b))=\pi(a\vee b)\wedge(a\vee\pi(a\vee b)\wedge\pi(b)))\leq\pi((a\vee b)\wedge a)\vee(\pi(a\vee b)\wedge\pi(b))=\pi((a\vee b)\wedge a)\vee\pi((a\vee b)\wedge b)=\pi(a)\vee\pi(b)$. Hence $\pi(a\vee b)\leq\pi(a)\vee\pi(b)$.

Let, conversely, every π on N has the property of the theorem. Then $a\wedge(b\vee(a\wedge c))=\pi_a(b\vee(c\wedge a))\leq\pi_a(b)\vee\pi_a(a\wedge c)=(a\wedge b)\vee(a\wedge c)$, from which the modularity of N follows by Lemma 3.

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