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ON THE BLASCHKE CONDITIONS OF (LP) FUNCTIONS AND NORMAL FUNCTIONS I

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1. Introduction

Let U be the unit disk, T the unit circle in the complex plane $\mathbb{C}$, and $S=\mathbb{C}\cup\{\infty\}$. (LP) denotes the class of all nonconstant analytic functions in U with radial limits 0 almost everywhere on T (see [7, p.185] and [6, p.79]). For a $\zeta\in S$ and for a function $f(z)$ defined in U, the set of zeros of $f(z)-\zeta$ is called the ζ-points of $f(z)$.

If ζ-points $\{z_n\}$ of $f(z)$ satisfy the Blaschke condition

$$\sum_{n=1}^{\infty}(1-|z_n|)<\infty,$$

then we say that ζ satisfies the Blaschke condition. A sequence of points satisfying the Blaschke condition is called a Blaschke sequence. A meromorphic function $f(z)$ is called nomal in U if the family $\{f(S(z))\,;\, S$ is a one-one conformal mapping of U onto itself$\}$ is normal.

Hwang [4] proposed the following conjectures;

Conjecture 1. If $f(z)$ is analytic in U and of class (LP), then there should be no nonzero numbers ζ satisfying the Blaschke condition.

Conjecture 2. If $f(z)$ is meromorphic in U and if there are three values satisfying the Blaschke condition, then $f(z)$ is normal in U.

In this paper, we disprove both conjectures by two simple examples. Finally, we refine the first example to get our main.

THEOREM. *For a complex number ζ and for a Blaschke sequence $\{z_n\}$, there exists a function analytic in U and of class (LP) whose ζ-points are exactly $\{z_n\}$, counted according to their multiplicities.*

2. EXAMPLE. Let $f(z)$ be a function analytic in U and of class (LP), and ζ a nonzero number. Then the function

$$F(z)=\zeta(1-e^{f(z)})\quad(z\in U)$$

is analytic in U and of class (LP) with empty ζ-points. Therefore ζ vacuously satisfies the Blaschke condition, and disproves Conjecture 1. We note that our main theorem covers the nonvacuous case.

3. EXAMPLE. $g(z)=(1-z)e^{(1+z)/(1-z)}$ $(z\in U)$.

Since $g(z)$ is a quotient of two bounded functions, $g(z)-\zeta$ is of bounded characteristic for arbitrary $\zeta\in\mathbb{C}$, hence ζ satisfies the Blaschke condition for arbitrary $\zeta\in\mathbb{C}$ [2, Theorem 2.3]. If γ be the upper semi-circle of $|z-\frac{1}{2}|=\frac{1}{2}$, then $g(z)$ has asymptotic value 0 along γ at $z=1$. Since $\lim_{r\to 1} g(r)=\infty$, $g(z)$ cannot be normal [5, Theorem 2]. This shows that Conjecture 2 is false.

4. LEMMA. *Let $B(z)$ be a nonconstant Blaschke product, $B^*(e^{i\theta})=\lim_{r\to 1} B(re^{i\theta})$, which exists almost every θ, and let*

$$E=\{e^{i\theta}\in T : |B^*(e^{i\theta})|=1 \text{ and } B^*(e^{i\theta})\neq 1\}.$$

For each $e^{i\theta}\in E$, let $t(\theta)$ be defined via $B^*(e^{i\theta})=e^{it(\theta)}$. Define a function $\phi : T\to [0,2\pi)$ by

$$(1)\quad \phi(e^{i\theta})=\begin{cases} t(\theta)\ (mod\ 2\pi) & if\ e^{i\theta}\in E \\ 0 & if\ e^{i\theta}\in T-E. \end{cases}$$

Then E is of measure 2π and $\phi(e^{i\theta})$ is a bounded measurable function.

PROOF. We need only to show the measureability of $\phi(e^{i\theta})$. The functions

$$B_n(e^{i\theta})=B\left(\frac{n-1}{n}e^{i\theta}\right),$$

defined on E is measurable, so is the limit function $B^*(e^{i\theta})$. But $B^*(e^{i\theta})=e^{it(\theta)}=e^{i\phi(e^{i\theta})}$ on E. Since the function $\log z$ on $\mathbb{C}-\{z\geq 0\}$ is continuous, $\phi(e^{i\theta})=-i\log(B^*(e^{i\theta}))$ is a measurable function on E. Thus $\phi(e^{i\theta})$ is measurable.

5. PROOF OF THEOREM. Let ζ be a complex number. Let $\{z_n\}$ be a Blaschke sequence, and $B(z)$ be the Blaschke product formed by $\{z_n\}$. By the help of Example 2 and Hwang's example in the proof of [4, Theorem 2], we may assume that $B(z)$ is nonconstant and ζ is nonzero. By Lemma,

$$E=\{e^{i\theta}\in T\ ;\ |B^*(e^{i\theta})|=1 \text{ and } B^*(e^{i\theta})\neq 1\}$$

is a set of measure 2π and the function $\phi(e^{i\theta})$ defined by (1) is a bounded measurable function. So $i\phi(e^{i\theta})\in L^1(T)$. If $P[i\phi](z)$ is the Poisson integral of $i\phi(e^{i\theta})$, then it is harmonic (so, continuous) in U and

$$\lim_{r\to 1} P[i\phi](re^{i\theta})=i\phi(e^{i\theta})=it(\theta)\ (\text{mod } 2\pi)$$

on a set, say E_1, of measure 2π. Let A be a set of first category with measure 2π (see [3, p.99]).

Then $A\cap E_1$ is also of first category with measure 2π. Theorefore by a theorem of F. Bagemihl and W. Seidel [1, Theorem 8.11], there exists an analytic function $h(z)$ in U with $\lim_{r\to 1} h(re^{i\theta})=it(\theta)$ (mod 2π) a.e. Then

$$F(z)=-\zeta B(z)e^{-h(z)}+\zeta \quad (z\in U)$$

is a nonconstant analytic function in U whose ζ-points coincides with the zeros of $B(z)$ with due count of multiplicity. Also $\lim_{r\to 1} F(re^{i\theta})=0$ a.e. ; so $f\in (LP)$. This completes the proof.

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REFERENCES

[1] E.F. Collingwood and A.J. Lohwater, *The theory of cluster sets,* Cambridge Univ. Press, New York, (1966).

[2] P.L. Duren, *Theory of H^p spaces,* Academic Press, New York, (1970).

[3] B.R. Gelbaum and J.M.N. Olmsted, *Counter-examples in analysis,* Holdenday, Inc. San Francisco, (1964).

[4] J.S. Hwang, *On the radial limits of analytic and meromorphic functions,* Trans. Amer. Math. Soc. 270 (1982), 341—348.

[5] O. Lehto and K.I. Virtanen, *Boundary behavior and normal meromorphic functions,* Acta Math. 97 (1957), 47—65.

[6] A.J. Lohwater, *On the radial limits of analytic functions,* Proc. Amer. Math. Soc. 6 (1955), 79—84.

[7] N. Lusin and J. Privaloff, *Sur l'unicité et la multiplicité des functions analytiques,* Ann. École Norm. 42 (1925), 143—191.