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ON CERTAIN LIPSCHITZIAN INVOLUTIONS IN BANACH SPACES

SEHIE PARK AND SANGSUK YIE

1. Introduction

In [3], [4], K. Goebel and E. Zlotkiewicz investigated conditions under which lipschitzian involutions or lipschitzian maps with nonexpansive square of a closed bounded convex subset X of a Banach space B have fixed points. A map $T : X \longrightarrow X$ is called an involution if $T^2 = I$, where I denotes the identity map, and a k-lipschitzian if $\|Tx - Ty\| \leq k\|x-y\|$ holds for all $x, y \in X$. A 1-lipschitzian map is said to be nonexpansive.

In the present paper, the main results of [3], [4] are so strengthened that some information concerning the geometric estimations of fixed points are given.

Our tool in this paper is the following in [7], which is a consequence of the well-known variational principle of Ekeland [1], [2] for approximate solutions of minimization problems.

THEOREM 0. *Let V be a complete metric space and $f : V \longrightarrow V$ be a map such that there exists an $L \in [0, 1)$ satisfying*

$$d(fx, f^2x) \leq Ld(x, fx) \text{ for any } x \in V.$$

If $F(x) = d(x, fx)$ on V is l. s. c., then

(1) $\lim f^n x = p$ exists for any $x \in V$,

$$d(f^n x, p) \leq \frac{L^n}{1-L} d(x, fx),$$

and p is a fixed point of f, and

(2) for any $u \in V$ and $\varepsilon > 0$ satisfying

$$F(u) \leq (1-L)\varepsilon,$$

f has a fixed point in $\bar{B}(u, \varepsilon)$. Further, if f is a quasi-lipschitzian with

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constant k, then either u is a fixed point of f or f has a fixed point in $\bar{B}(u,\varepsilon)\backslash B(u,s)$ *where* $s=d(u,fu)(1+k)^{-1}$.

Note that $\bar{B}(u,\varepsilon)$ denotes the closed ball with center u and radius ε, and $B(u,\varepsilon)$ the corresponding open ball.

A map $f: V \longrightarrow V$ is called a quasi-lipschitzian with constant k if $\|fx-fp\| \le k\|x-p\|$ holds for all $x \in V$ and for every fixed point p of f.

2. Main Results

The modulus of convexity of the space B is the function $\delta: [0,2] \longrightarrow [0,1]$ defined by the following formula

$$\delta(\varepsilon)=\inf\{1-\|\frac{x+y}{2}\| : x, y \in \bar{B}(0,1),\ \|x-y\| \ge \varepsilon\}.$$

Note that the function $\delta(\varepsilon)$ is nonincreasing and convex.

Moreover, for any $x, y \in \bar{B}(0,r)$ and any a such that $0 \le a \le 2r$ and $\|x-y\| \ge a$, we have

$$\|\frac{x+y}{2}\| \le (1-\delta(\frac{a}{r}))r \qquad [3].$$

Now we have our first result:

THEOREM 1. *Let X be a closed convex subset of a Banach space B and* $T: X \longrightarrow X$ *a k-lipschitzian involution. If* $L: =k(1-\delta(2/k))/2<1$, *then for any* $u \in X$ *and* $\varepsilon>0$ *satisfying*

$$\|u-Tu\| \le (1-L)\varepsilon,$$

either u is a fixed point of T or there is a fixed point of T in $\bar{B}(u,\varepsilon/2) \cap X \backslash B(u,s)$ *where* $s=\|u-Tu\|(k+3)^{-1}$.

Proof. For any $x \in X$,

$$\begin{aligned}\|T\left(\frac{x+Tx}{2}\right)-x\| &= \|T\left(\frac{x+Tx}{2}\right)-T^2x\| \\ &\le k\|\frac{x+Tx}{2}-Tx\| \\ &= \frac{k}{2}\|x-Tx\| \\ \|T\left(\frac{x+Tx}{2}\right)-Tx\| &\le k\|\frac{x+Tx}{2}-x\| \\ &= \frac{k}{2}\|x-Tx\|.\end{aligned}$$

Thus, by the property of modulus of convexity, we have

$$\left\|\frac{x+Tx}{2}-T\left(\frac{x+Tx}{2}\right)\right\|\le\left(1-\delta\left(\frac{2}{k}\right)\right)\frac{k}{2}\|x-Tx\|.$$

Now if we put $G=\frac{1}{2}(I+T)$, then

$$\begin{aligned}\|Gx-G^2x\| &=\left\|\frac{Gx+TGx}{2}-Gx\right\|\\ &=\frac{1}{2}\|TGx-Gx\|\\ &\le\frac{1}{2}\left(1-\delta\left(\frac{2}{k}\right)\right)\frac{k}{2}\|x-Tx\|\\ &=\left(1-\delta\left(\frac{2}{k}\right)\right)\frac{k}{2}\|x-Gx\|\\ &=L\|x-Gx\|.\end{aligned}$$

Therefore, by Theorem 0(1), $\lim G^n x=p$ exists for $x\in X$, and $p\in\mathrm{Fix}\,G=\mathrm{Fix}\,T$, the fixed point set. Since T is a k-lipschitzian, G is a $(k+1)/2$-lipschitzian and quasi-lipschitzian. Therefore, by Theorem 0(2), for any $u\in X$ with $\|u-Tu\|\le(1-L)\varepsilon$, we have $\|u-Gu\|=\|u-Tu\|/2\le(1-L)\varepsilon/2$. Hence, u is a fixed point of G or there is a fixed point of G in $\bar{B}(u,\varepsilon/2)\cap X\backslash B(u,s)$ where $s=\|u-Gu\|/(1+(k+1)/2)=\|u-Tu\|/(k+3)$. This completes our proof.

COROLLARY 1. [3, Theorem 1]. *Let X be a closed convex subset of a Banach space B and $T: X\longrightarrow X$ a k-lipschitzian involution such that $k(1-\delta(2/k))/2<1$. Then T has at least one fixed point.*

COROLLARY 2. *Let X be a closed convex subset of a Banach space B and $T: X\longrightarrow X$ a k-lipschitzian involution. If $0\le k<2$, then for any $u\in X$ and $\varepsilon>0$ satisfying*

$$\|u-Tu\|\le\left(1-\frac{k}{2}\right)\varepsilon$$

the conclusion of Theorem 1 holds.

Proof. Let $L=k/2$ and $G=\frac{1}{2}(I+T)$. Then $L<1$, and

$$\begin{aligned}\|Gx-G^2x\| &\le\left(1-\delta\left(\frac{2}{k}\right)\right)\frac{k}{2}\|x-Gx\|\\ &\le\frac{k}{2}\|x-Gx\|\\ &=L\|x-Gx\|.\end{aligned}$$

Thus, by Theorem 1, we have the same conclusion to Theorem 1.

Corollary 2 improves [4, Theorem 1].

The characteristic of convexity of the space B is the number $\varepsilon_0=\sup\{\varepsilon : \delta(\varepsilon)=0\}$.

Some of Banach spaces can be fully characterized by the number ε_0 and the modulus of convexity. The following facts are known [3]

(1) If $\varepsilon_0<1$, then B has normal structure,
(2) B is uniformly non-square iff $\varepsilon_0<2$, and
(3) B is strictly convex iff $\delta(2)=1$.

THEOREM 2. *Let X be a closed convex bounded subset of a Banach space B with $\varepsilon_0<1$ and $\delta(2)=1$, and $T: X\longrightarrow X$ a k-lipschitzian map such that T^2 is nonexpansive. If $L:=k(1-\delta(2/k))/2<1$, then the conclusion of Theorem 1 holds.*

Proof. Since $\varepsilon_0<1$, B is uniformly non-square and in view of [5], it is reflexive, and moreover it has normal structure. Since T^2 is nonexpansive, by Kirk's fixed point theorem [6], the set $C^*=\{x : T^2x=x\}$ is nonempty. $\delta(2)=1$ means the strict convexity of B and implies that C^* is convex. Obviously we have $T(C^*)=C^*$ and $T^2=I$ on C^*. Hence, using Corollary 1 for the restriction of T on C^*, we can apply Theorem 1.

Theorem 2 improves [3, Theorem 2].

THEOREM 3. *Let X be a closed convex subset of a uniformly convex Banach space B and $T: X\longrightarrow X$ a k-lipschitzian involution. If $L:=k\delta^{-1}(1-1/k)/4<1$, then for any $u\in X$ and $\varepsilon>0$ satisfying $\|u-Tu\|<(1-L)\varepsilon$, either u is a fixed point of T or T has a fixed point in $\bar{B}(u, \varepsilon/2)\cap X\backslash B(u, s)$ where $s=\|u-Tu\|(k+3)^{-1}$.*

Proof. Let $G=(I+T)/2$ and let for $x\in X$, $y=Gx$ and $z=Ty$. Then

$$\begin{aligned}\|z-x\|&=\|Ty-x\|=\|Ty-T^2x\|\\&\leq k\|y-Tx\|=k\|Gx-Tx\|\\&=k\left\|\frac{x+Tx}{2}-Tx\right\|=\frac{k}{2}\|x-Tx\|\end{aligned}$$

$$\begin{aligned}\|(2y-z)-x\|&=\|2Gx-Ty-x\|=\|x+Tx-Ty-x\|\\&=\|Tx-Ty\|\leq k\|x-y\|=k\|x-Gx\|\\&=k\left\|x-\frac{x+Tx}{2}\right\|=\frac{k}{2}\|x-Tx\|\end{aligned}$$

and

$$\left\|\frac{z+(2y-z)}{2}-x\right\|=\|y-x\|=\|x-Gx\|$$
$$=\frac{1}{2}\|x-Tx\|.$$

Thus, by the property of modulus of convexity, we have

$$\|z-(2y-z)\|\leq\frac{k}{2}\delta^{-1}\left(1-\frac{1}{k}\right)\|x-Tx\|.$$

But since

$$\|z-(2y-z)\|=2\|y-z\|=2\|Gx-Ty\|$$
$$=4\|Gx-G^2x\|$$

and
$$\|x-Tx\|=2\|x-Gx\|,$$

we have

$$4\|Gx-G^2x\|\leq k\delta^{-1}\left(1-\frac{1}{k}\right)\|x-Tx\|$$

i. e.,
$$\|Gx-G^2x\|\leq\frac{k}{4}\delta^{-1}\left(1-\frac{1}{k}\right)\|x-Gx\|$$
$$=L\|x-Gx\|,$$

and
$$\|x-Gx\|\leq(1-L)\varepsilon/2.$$

Thus G satisfies all the hypothesis of Theorem 0, and we conclude our result.

Theorem 3 strengthens [4, Theorem 2].

THEOREM 4. *Let X be a closed bounded convex subset of a uniformly convex Banach space B, and $T: X \longrightarrow X$ a k-lipschitzian map such that T^2 is nonexpansive. If $L := k\delta^{-1}(1-1/k)/4<1$, then the conclusion of Theorem 3 holds.*

Proof. Since every uniformly convex Banach space is strictly convex, reflexive and has normal structure, by Theorem 2 and Theorem 3, we obtain our result.

Theorem 4 strengthens [4, Theorem 3].

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Seoul National University
Seoul 151, Korea
and
Soong Jun University
Seoul 151, Korea