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EPIMORPHISMS OF NONCOMMUTATIVE C^*-ALGEBRAS

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1. Introduction

It was proved in Esterle [4] that every epimorphism from $C(X)$ onto a Banach algebra is continuous. Here $C(X)$ denotes the commutative C^*-algebra of continuous functions on a compact Hausdorff space X. Then Laursen has shown in [6] that an epimorphism from a C^*-algebra onto a commutative Banach algebra is necessarily continuous. However, it remains unknown whether every epimorphism of a C^*-algebra with noncommutative range is continuous.

In this note we study some properties of epimorphisms of C^*-algebras and give sufficient conditions for continuity of epimorphisms with noncommutative range. Let A be a C^*-algebra and B a Banach algebra. We show that for every epimorphism $\theta : A \to B$ the image of the closure of the kernel of θ coincides with the radical of B. Thus an epimorphism $\theta : A \to B$ is continuous if the radical of B is commutative. This result slightly generalizes the Laursen's result. Also it is shown that an epimorphism $\theta : A \to B$ is continuous if the radical R of B satisfies the condition $\cap (R^n)^- = \{0\}$. This result generalizes corollaries 3.4 and 3.5 in [2] for epimorphisms of C^*-algebras.

2. Preliminaries

Let A and B be Banach algebras. By a homomorphism $\theta : A \to B$ we mean a multiplicative linear map which maps A into B, and an epimorphism means a surjective homomorphism. For a homomorphism $\theta : A \to B$ the separating space of θ is a linear subspace of B defined by

$$\mathcal{S}(\theta) = \{ b \in B: \text{ there is a sequence } a_n \to 0 \text{ in } A \text{ with } \theta(a_n) \to b \}.$$

A homomorphism $\theta : A \to B$ is continuous if and only if $\mathcal{S}(\theta) = \{0\}$ by

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the closed graph theorem, and it can be easily shown that $\mathcal{S}(\theta)$ is a closed two-sided ideal of B if the range of θ is dense in B [10].

The radical of a Banach algebra is the intersection of all maximal modular left ideals of the algebra, and a Banach algebra is called semisimple if the radical of the algebra contains only the zero element. If a Banach algebra has no maximal modular left ideal, the radical of the algebra is defined to be the algebra itself, in this case the Banach algebra is called a radical algebra. The radical of a Banach algebra is a closed two-sided ideal, and it is itself a radical algebra since the radical of a closed two sided ideal of a Banach algebra is the intersection of the ideal and the algebra [1: Coro. 24. 20].

An element a of a C^*-algebra is called self-adjoint if $a=a^*$, and a subalgebra is called self-adjoint if it is closed under involution. Every closed two-sided ideal of a C^*-algebra is known to be selfadjoint, and the quotient algebra of a closed two-sided ideal is also self-adjoint hence such a quotient algebra is itself a C^*-algebra with the quotient norm. It is well-known that a C^*-algebra is semisimple.

The most important positive result on the continuity of epimorphisms on which our results depend is the following theorem due to Johnson [5].

THEOREM (Johnson) *Every epimorphism from a Banach algebra onto a semisimple Banach algebra is continuous.*

3. Epimorphisms

LEMMA 1. *Let A and B be the Banach algebras and $\theta : A \to B$ be an epimorphism. Then for each maximal modular left ideal M of B the inverse image of M is a maximal modular left ideal of A containing the kernel of θ.*

Proof. Let M be a maximal modular left ideal of B. Then the inverse image $\theta^{-1}(M)$ is clearly a modular left ideal of B containing the kernel of θ. Let u be a right modular unit for $\theta^{-1}(M)$ and let M' be a proper left ideal of A containing $\theta^{-1}(M)$. Then $\theta(M')$ is a modular left ideal of B containing M. Suppose that $\theta(M')=B$. Then $\theta(u)\in\theta(M')$ and hence there is an element $a\in M'$ with $\theta(u)=\theta(a)$. Hence u must belong to M', which is impossible since it is a right modular unit for M'. Thus we have $\theta(M')\neq B$, hence by maximality of $M, \theta(M')=M$ and thus $M'=\theta^{-1}(M)$. Therefore $\theta^{-1}(M)$ is maximal.

The context of the following lemma is included in the proof of Proposition 25. 10 of [1]. But we include another proof of the lemma since it is an easy consequence of Lemma 1.

LEMMA 2. *Let A and B be Banach algebras and $\theta : A \to B$ be an epimorphism with the kernel K. Then $\theta(\bar{K})$ is contained in the radical R of B.*

Proof. For each maximal modular left ideal M of B the closure $\bar{K}$ of the kernel is contained in $\theta^{-1}(M)$ since $\theta^{-1}(M)$ is closed and contains the kernel K. Hence we have

$$\bar{K} \subset \cap \{\theta^{-1}(M) : M \text{ is a maximal modular left ideal of } B\}.$$

Since $\theta(\cap \theta^{-1}(M)) \subset \cap \theta(\theta^{-1}(M))$ and $\theta(\theta^{-1}(M)) = M$, we have

$$\theta(\bar{K}) \subset \theta(\cap \theta^{-1}(M)) \subset \cap M = R$$

where intersection is taken over all maximal modular left ideals M of B.

In the following lemma we prove that for every epimorphism θ of a C^*-algebra with the kernel K, $\theta(\bar{K})$ is closed without assuming the continuity of θ.

LEMMA 3. *Let A be a C^*-algebra and B a Banach algebra. Then for each epimorphism $\theta : A \to B$ the image of the closure of the kernel of θ is closed in B.*

Proof. Let $\bar{K}$ denote the closure of the kernel of θ and let $\pi : A \to A/\bar{K}$ be the quotient map. Now, we define a map

$$\bar{\theta} : B \to A/\bar{K} \text{ by } \bar{\theta}(b) = a + \bar{K}$$

where a is an element of A with $\theta(a) = b$. The map $\bar{\theta}$ is well-defined and clearly it is an epimorphism from B onto the C^*-algebra $A/\bar{K}$. By Johnson's theorem the epimorphism $\bar{\theta}$ is continuous and it has the closed kernel. To complete the proof it is enough to show that $\theta(\bar{K}) = \mathrm{Ker}(\bar{\theta})$ where $\ker(\bar{\theta})$ denotes the kernel of $\bar{\theta}$

If $b \in \theta(\bar{K})$ there is an $a \in \bar{K}$ with $\theta(a) = b$. Thus $\bar{\theta}(b) = a + \bar{K} = \bar{K}$, which implies that b belongs to $\ker(\bar{\theta})$. Conversely, let b be an element of $\ker(\bar{\theta})$, then $\bar{\theta}(b) = \bar{K}$. Let $a \in A$ with $\theta(a) = b$. By the definition of $\bar{\theta}$ we have $a + \bar{K} = \theta(b) = \bar{K}$. Hence $a \in \bar{K}$ and $b \in \theta(\bar{K})$.

THEOREM 4. *Let A be a C^*-algebra and B a Banach algebra with the radical R. For every epimorphism $\theta : A \to B$ we have $\theta(\bar{K}) = R$.*

Proof. It is enough to show that $R \subset \theta(\bar{K})$ since we have already shown that $\theta(\bar{K}) \subset R$. Since $\theta(\bar{K})$ is a closed two-sided ideal of B the quotient algebra $B/\theta(\bar{K})$ is a Banach algebra. Let $\bar{\theta} : B \to A/\bar{K}$ be the epimorphism defined in Lemma 3 and let $\phi : B \to B/\theta(\bar{K})$ be the quotient map. Since the kernel of $\bar{\theta}$ and $\theta(\bar{K})$ coincide there exists a continuous isomorphism

$$\hat{\theta} : B/\theta(\bar{K}) \to A/\bar{K}$$

such that $\bar{\theta} = \hat{\theta} \circ \phi$. Since $\hat{\theta}$ maps $B/\theta(\bar{K})$ onto the C^*-algebra $A/\bar{K}$ the Banach algebra $B/\theta(\bar{K})$ is semisimple.
On the other hand, by Lemma 1

$$R = \cap \{ M' : M' \text{ is a maximal modular left ideal of } B \}$$
$$\subset \cap \{ \phi^{-1}(M) : M \text{ is a maximal modular left ideal of } B/\theta(\bar{K}) \}$$

Thus we have

$$\phi(R) \subset \cap \{ M : M \text{ is a maximal modular left ideal of } B/\theta(\bar{K}) \}$$
$$= \text{the radical of } B/\theta(\bar{K}) = \{0\}$$

since $B/\theta(\bar{K})$ is semisimple. Hence

$$R \subset \ker(\phi) = \theta(\bar{K}).$$

REMARK. For an epimorphism $\theta : A \to B$ from a C^*-algebra onto a Banach algebra B it is known that $R = \mathfrak{S}(\theta)$ where $\mathfrak{S}(\theta)$ is the separating space of θ[8 : Thm 4.1], [3 : Coro. 2.2]. Hence we have $\theta(\bar{K}) = R = \mathfrak{S}(\theta)$.

COROLLARY 5. *Let A and B be as in Theorem 4. If the radical R of B is commutative, then an epimorphism $\theta : A \to B$ is continuous.*

Proof. Let $\hat{\theta} : \bar{K} \to R$ be the restriction of an epimorphism $\theta : A \to B$ to the closure of the kernel K of θ. Then $\hat{\theta}$ is an epimorphism of a C^*-algebra with commutative range, hence it is continuous by Laursen's result. Consequently $\hat{\theta}$ has the closed kernel and $K = \bar{K}$. Therefore we have $\mathfrak{S}(\theta) = \theta(\bar{K}) = \{0\}$.

THEOREM 6. *Let A be a C^*-algebra and B a Banach algebra with the radical R. If $\cap (R^n)^- = \{0\}$, then every epimorphism $\theta : A \to B$ is continuous. Here, $(R^n)^-$ denotes the closure of R^n.*

Proof. Let $\bar{K}$ be the closure of the kernel of θ. Since the restriction $\theta : \bar{K} \to R$ is an epimorphism of a C^*-algebra, to prove the continuity of θ it is enough to show that θ is continuous on each commutative

C^*-subalgebra generated by a self-adjoint element of $\bar{K}$ [8 : Coro. 4. 3].

Let $a \in \bar{K}$ be a self-adjoint element, $C^*(a)$ denote the C^*-algebra generated by a and D be the closure of $\theta(C^*(a))$. Since D is a closed subalgebra of the radical algebra R, it is a radical algeba.

Suppose that the restricted homomorphism $\theta : C^*(a) \rightarrow D$ is discontinuous for some self-adjoint element a of $\bar{K}$. Applying Theorem 4. 3 of [9] we see that for every element x of $C^*(a)$ $\theta(x)$ is not nilpotent and

$$(\theta(x)D)^- = (\theta(x)^n D)^-$$

for each positive integer n. Suppose that there is an element in $C^*(a)$ with $\theta(y) \neq 0$. Since $\theta(y) \in D$ we have

$$\theta(y)^2 \in (\theta(y)^n D)^- \subset (R^{n+1})^-$$

for each positive integer n and hence

$$\theta(y)^2 \in \bigcap_{n=1}^{\infty} (R^n)^- = \{0\},$$

which implies that $\theta(y)^2 = 0$. This is contrary to $\theta(y)$ being not nilpotent, and θ must be continuous on $C^*(a)$ for each self-adjoint element a in $\bar{K}$. Therefore the epimorphism $\theta : \bar{K} \rightarrow R$ is continuous and we have $\mathcal{S}(\theta) = \{0\}$.

If B is a Banach algebra satisfying the descending chain condition for left (or right) ideals, then the radical R of B is nilpotent, that is, there is a positive integer n such that $R^n = \{0\}$. (See e. g. [7; p. 120]). Consequently we have the following corollary to Theorem 6.

COROLLARY 7. *If A is a C^*-algebra and B is a Banach algebra satisfying the descending chain condition for left ideals, then every epimorphism $\theta : A \rightarrow B$ is continuous.*

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