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SUFFICIENT CONDITIONS FOR LOCAL SOLVABILITY OF NONSOLVABLE PSEUDODIFFERENTIAL OPERATORS

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Introduction

Let D be a pseudodifferential operator acting upon $C^\infty(\Omega; H^{\pm\infty}(R^n))$. In this paper we consider the existence of local solution of the equation

(0. 1) $\qquad D_u=f$

for a given $f\in C_0^\infty(\Omega\times R^n)$, i. e., we investigate what conditions should be imposed on the function f for the existence of a C^1 solution u of the equation (0. 1) in some neighborhood of the origin. We consider only the case when $D=d_t+d_tB(t, D_x)$ is a nonsolvable operator in a neighborhood of the origin. The main result of this paper is given in Prop. 3. 1.

1. Preliminaries

Let R^ν (resp. R^n) be a ν-dimensional (resp. n-dimensional) Euclidean space. Throughout this paper, we shall denote by Ω an open subset of R^n and by R_n the dual of R^n. For any real number we denote by $H^s=H^s(R^n)$ the standard Sobolev space on R^n, i. e., the space of tempered disributions u in R^n whose Fourier transform $\hat{u}$ is a measurable function in R_n, satisfying

$$\|u\|_s=(2\pi)^{-n}\left(\int(1+|\xi|^2)^s|\hat{u}(\xi)|^2d\xi\right)^{\frac{1}{2}}<+\infty.$$

Starting with H^s we build the following spaces

$$H^{-\infty}=\bigcup_{s\in R}H^s,\quad H^{+\infty}=\bigcap_{s\in R}H^s.$$

For any real s, let E^s denote the subspace consisting of the generalized functions u whose Fourier transform $\hat{u}$ is a measurable function in R_n satisfying

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$$|||u|||_s=(2\pi)^{-n}\left(\int e^{2s|\xi|}|\hat{u}(\xi)|^2d\xi\right)^{\frac{1}{2}}<+\infty.$$

As with the Sobolev spaces, we form the union and intersection of the space E^s, but for s going to zero:

$$E^{0+}=\bigcup_{s>0}E^s, \quad E^{0-}=\bigcap_{s>0}E^{-s}.$$

Let $t=(t, \ldots, t_\nu)$ denote the variable point in an open set $\Omega\subset R^\nu$. Let E be any one of the spaces $H^{\pm\infty}, E^{0\pm}$. If p is any integer such that $0\leqq p\leqq\nu$, we denote by $\Lambda^pC^\infty(\Omega;E)$ the spaces of C^∞ p-form valued in the space E. Thus to say that u belong to $\Lambda^pC^\infty(\Omega;E)$ is the same as the saying that

$$u(t,x)=\sum_{|J|=p}u_Jdt_J$$

where J is an ordered multi-index $(j_1,\cdots,j_p)$ of integers such that $1\leqq j_1<\cdots<j_p\leqq\nu$, $|J|$ its length, here equal to p, $dt_J=dt_{j_1}\Lambda\ldots\Lambda dt_{j_p}$ and u_J are C^∞ functions from Ω to E.

Now we consider a C^∞one form in Ω, depending on the parameter ξ of R_n;

$$b(t,\xi)=\sum_{j=1}^{\nu}b_j(t,\xi)dt_j.$$

We assume that the one form $b(t,\xi)$ is exact in Ω. Thus there exists a primitive B of b such that $b(t,\xi)=d_tB(t,\xi)$. We also assume that

(1) $B(t,\xi)$ is real valued and positive homogeneous of degree one with respect to ξ, and

(2) $B(t,\xi)$ is a C^∞ function of t in Ω with values in $C^1(R_n\backslash 0)$.

Under these assumptions $b_j(t,D_x)$ defines naturally a pseudodifferential operator

$$b_j(t,D_x)u(t,x)=(2\pi)^{-n}\int e^{ix\cdot\xi}b_j(t,\xi)\hat{u}(t,\xi)d\xi.$$

Here $\hat{u}$ denotes the Fourier transform with respect to x. We form a pseudodifferential operator

$$D=d_t+b(t,D_x)\Lambda.$$

For each $p=0,1,\ldots,\nu-1$, it defines a linear operator

$$D^p:\Lambda^pC^\infty(\Omega;E)\longrightarrow\Lambda^{p+1}C^\infty(\Omega;E).$$

We set $D^\nu=0$. Then obviously we have

$$D^2=D^{p+1}\circ D^p=0$$

for any $p=0, 1, \cdots, \nu-1$. We note that $\hat{D}=e^{-B(t,\xi)}d_t e^{B(t,\xi)}$. It is evident that $\hat{D}$, hence also D, generates a complex.

We concern the equation

(1.1) $$Du=f,$$

where $f \in \Lambda^{p+1}C^\infty(\Omega;E)$. the By the Fourier transform with respect to x we see that (1.1) is equivalent to

(1.2) $$d_t(e^B\hat{u})=e^B f \text{ (for a. e. } \xi \text{ in } R_n)$$

We denote by $\mathscr{B}_D{}^pC^\infty(\Omega;E)$ the space of elements f of $\Lambda^pC^\infty(\Omega;E)$ which satisfying the compatibility condition; namely,

(1.3) for a. e. ξ in R_n, the p-form $e^{B(t,\xi)}\hat{f}(t,\xi)$ is a coboundary

2. Property (ψ) and the statement of the solvability results.

F. Treves [3] found the necessary and sufficiant condition for the solvability of the equation (1.1), which is a natural generalization of the condition (P) for a single linear partial differential operator. We will state the property (ψ) and the solvability results in F. Treves [3].

We consider the complex

$$\cdots \to \Lambda^pC^\infty(\Omega;E) \xrightarrow{D^p} \Lambda^{p+1}C^\infty(\Omega;E) \to \cdots.$$

Let Ω' be a nonempty open subset of Ω, possibly equal to Ω. Let U, V be two open subsets of Ω such that $U \subset \Omega' \cap V$. For any $\xi \in R_n$ and any real r we write

$$U(\xi, r)=\{t \in U; B(t,\xi)<r\}$$

and similarly with V substituted for U.

We consider the natural homomorphisms

(2.1) $$H_p(U(\xi,r)) \begin{array}{l} \xrightarrow{i_p} H_p(\Omega') \\ \xrightarrow[j_p]{} H_p(V(\xi,r)) \end{array}$$

where H_p's stand for the p-th homology groups.

DEFINITION 2.1. We say that the system D has property (ψ), in dimension p, in Ω' relative to Ω, if to every open subset $U \subset \Omega'$ there exists an open set $V \subset\subset \Omega$ containing U such that, given any ξ in R_n and real number r,

(2.2) $$\text{Ker } i_p \subset \text{Ker } j_p$$

We say that D has property (ψ), in dimension p in Ω if it has it in

Ω relative to itself, i.e., $\Omega=\Omega'$.

THEOREM 2.1. *Suppose that D does not have property* (ψ), *in dimension p, in Ω' relative to Ω. Then there is an element f of $\mathcal{B}_D{}^{p+1}C^\infty(\Omega; H^{+\infty})$ and a relatively compact subset U of Ω' such that*

$$Du=f \text{ in } U$$

has no solution u in $\Lambda^p C^\infty(U, H^{-\infty})$

THEOREM 2.2. *Suppose that the system D has property* (ψ), *in dimension p, in Ω' relative to Ω. Let E be any one of $H^{\pm\infty}$, $E^{0\pm}$. Then, given any relatively compact open subset U of Ω' and any element f in $\mathcal{B}_D{}^{p+1}C^\infty(\Omega; E)$, the equation*

$$Du=f \text{ in } U$$

has a solution u in $\Lambda^p C^\infty(U; E)$.

For the proofs of Theorem 2.1 and Theorem 2.2 see Section II.3 in [3].

REMARK 2.1. When Ω' is homologically trivial in dimension p, the property (2.2) can be stated in a simpler manner. Thus assume that $H_0(\Omega')=\mathbf{C}$ and $H_p(\Omega')=0$ $(p>0)$. Then (2.2) has the following mea ning: If $p=0$,

(2.3) any two points in $U(\xi, r)$ can be joined by a continuons path contained in $V(\xi, r)$, whereas, if $p>0$,

(2.4) every p-cycle in $U(\xi, r)$ is homologeous to zero in $V(\xi, r)$.

PROPOSITION 2.1. *The system D has property* (ψ), *in dimension $\nu-1$, in Ω' relative to Ω if and only if any one of the following equivalent properties holds, for any ξ in $\boldsymbol{R}_n$ and any r in $\boldsymbol{R}$:*

(2.5) *The natural homomorphism $H_{\nu-1}(\Omega'(\xi, r))\to H_{\nu-1}(\Omega')$ is injective.*

(2.6) *The natural homomorphism $H^{\nu-1}(\Omega') \to H^{\nu-1}(\Omega'(\xi, r))$ is surjective.*

(2.7) *No connected component of $\Omega'\backslash\Omega'(\xi, r)$ is compact*

For the proof see [3].

EXAMPLE 2.1. Let $B(t, \xi)=-(t_1{}^2+t_2{}^2)|\xi|$, where $(t_1, t_2)\in\Omega=(-T_1, T_1)\times(-T_2, T_2)$ and $\xi\in\boldsymbol{R}_n$, Then $\boldsymbol{D}=d_t+d_tB(t, D_x)\Lambda$ does not satisfy the property (2.3) when $p=0$ and hence D is a nonsolvable operator in dimension 0, in Ω. Also D does not satisfy the property (2.7) when $p=1$, and therefore D is a nonsolvable operator in dimension 1, in Ω.

REMARK 2.2. When $\nu=1$, there is only one case: $p=0=\nu-1$, and (2.2) is equivalent to (2.7). Let us take $\Omega'=\Omega$ to be an interval. Then (2.2) is equivalent to (2.3). When $\nu=1$, one deals with a single operator $D=\partial/\partial t+b(t, D_x)$, where $b(t,\xi)=\partial B(t,\xi)/\partial t$. It is seen at once that the validity of (ψ) in Ω (in dimension zero) is equivalent to the following property:

(2.8) For all $\xi\in R_n$ if $b(t^0,\xi)>0$ for some t^0 in Ω, then $b(t,\xi)\geqq 0$ for every t in $\Omega, t>t^0$.

EXAMPLE 2.2. Let $B(t,\xi)=-t^2|\xi|$ and hence $D=\partial/\partial t+b(t, D_x)$. Let Ω be an interval containing the origin. Then $b(t,\xi)$ does not satisfy conditon (2.8) and hence $\boldsymbol{D}$ is a nonsolvable operator.

3. Sufficient conditions for the solvability of nonsolvable operators

In this section we concern nonsolvable operators. First we deal a single operator $D=\partial/\partial t+b(t, D_x)$, where $b(t,\xi)=\partial B(t,\xi)/\partial t$. We will make the assumptions for $B(t,\xi)$ as follows:

(3.1) For some fixed $\xi\in R_n\setminus\{0\}$, $B(t,\xi)$ has a local maximum at $t=0$, in which case the function $B(t,\xi)$ of t is decreasing in $(0, T)$ and increasing in $(-T, 0)$. Let $V_1=\{\xi\in R_n\setminus 0 : B(t,\xi)$ is decreasing in $(0, T)$ and increasing in $(-T,0)$ with respect to $t\}$. Then V_1 is a cone since $B(t,\xi)$ is positive homogeneous of degree 1 with repect to ξ.

(3.2) For any fixed $\xi\in R_n\setminus V_1$, $B(t,\xi)$ is a monotone function of t in $(-T, T)$ or it has a local minimum at $t=0$, in which case $B(t,\xi)$ is increasing in $(0, T)$ and decreasing in $(-T,0)$.

If $B(t,\xi)$ satisfies the condition (3.1), then, from Remark 2.2, we see that the operator $\partial/\partial t+b(t, D_x)$, where $\partial B(t,\xi)/\partial t=b(t,\xi)$, is a nonsolvable operator.

PROPOSITION 3.1. *Let* $B(t,\xi)$ *satisfy the above hypothesis* (3.1) *and* (3.2). *Let* $f\in C_0^\infty((-T, T)\times R^n)$ *and*

$$Kf(x)=(2\pi)^{-n}\int_{S=-T}^{T}\int_{R_n} e^{ix.\xi-B(0,\xi)+B(s,\xi)}\chi_{V_1}(\xi)\hat{f}(s,\xi)ds\ d\xi$$

be real analytic. Then

(3.3) $$Du=f \ \textit{in} \ (-T, T)\times R_n$$

has a C^1 *solution.*

Proof. If we take a Fourier transform (w. r. to x) of the equation (3. 3), then we have

(3.4) $$\frac{\partial \hat{u}}{\partial t}+b(t,\xi)\hat{u}=\hat{f}(t,\xi).$$

We have a formal solution of the equation (3.4):

$$\hat{u}(t,\xi)=\int_{t_0}^{t} e^{-B(t,\xi)+B(s,\xi)}\hat{f}(s,\xi)\,ds,$$

where t_0 depends upon ξ. Let

$V_2=\{\xi \in R_n\backslash V_1 : B(t,\xi)$ is a monotone increasing function of t in $(-T, T)\}$.

Then for $\xi \in V_2$, we have an integral representation of a solution of the equation (3.4) as follows:

$$\hat{u}(t,\xi)=\int_{-T}^{t} e^{-B(t,\xi)+B(s,\xi)}\hat{f}(s,\xi)\,ds.$$

In this case $\chi_{V_2}(\xi)\hat{u}(t,\xi)$ is a rapidly decreasing function and $\mathcal{F}^{-1}(\chi_{V_2}\hat{u})$ is a C^∞ function of (t,x)

Let $V_3=\{\xi \in R_n\backslash(V_1\cup V_2) : B(t,\xi)$ is decreasing in $(-T, T)\}$. Then for $\xi \in V_3$, we have an integral representation of a solution of the equation (3.4) as follows:

$$\hat{u}(t,\xi)=-\int_{t}^{T} e^{-B(t,\xi)+B(s,\xi)}\hat{f}(s,\xi)\,ds.$$

In this case $\chi_{V_3}(\xi)\hat{u}(t,\xi)$ is a rapidly decreasing function and $\mathcal{F}^{-1}(\chi_{V_3}\hat{u})$ is a C^∞ function of (t,x).

Let $V_4=\{\xi \in R_n\backslash(V_1\cup V_2\cup V_3) : B(t,\xi)$ is increasing in $(0, T)$ and decreasing in $(-T, 0)\}$. Then for $\xi \in V_4$, we have a solotion of the equation (3.4) as follows:

$$\hat{u}(t,\xi)=\int_{0}^{t} e^{-B(t,\xi)+B(s,\xi)}\hat{f}(s,\xi)\,ds.$$

In this case $\chi_{V_4}(\xi)\hat{u}(t,\xi)$ is a rapidly decreasing function and $\mathcal{F}^{-1}(\chi_{V_4}\hat{u})$ is a C^∞ function of (t,x).

Let V_1 be a set defined in the condition (3.1) . Then for $\xi \in V_1$, if $t>0$, then we have a solution of the equation (3.4) as follows:

$$\hat{u}(t, \xi) = -\int_t^T e^{-B(t,\xi)+B(s,\xi)} \hat{f}(s, \xi) ds.$$

and if $t<0$, we have a solution of the equation (3.4) as follows:

$$\hat{u}(t, \xi) = \int_{-T}^t e^{-B(t,\xi)+B(s,\xi)} \hat{f}(s, \xi) ds$$

In this case

$$\hat{u}(+0, \xi) - \hat{u}(-0, \xi) = -\int_{-T}^T e^{-B(0,\xi)+B(s,\xi)} f(s, \xi) d\xi,$$

which means that $\hat{u}(t, \xi)$ is not continuous at $t=0$ when $\xi \in V_1$. If we denote $\frac{\partial}{\partial t}$ the derivative in the distribution sense and $\left[\frac{\partial}{\partial t}\right]$ the classical derivative in $t \neq 0$, then for $\xi \in V_1$

$$\frac{\partial u}{\partial t} = \left[\frac{\partial u}{\partial t}\right] + \delta(t) (\hat{u}(+0, \xi) - \hat{u}(-0, \xi))$$

Therefore for all $\xi \in R_n$, $\hat{u}(t, \xi)$ is a solution of the following equation

(3.5) $$\frac{\partial \hat{u}}{\partial t} + b(t, \xi)\hat{u} = \left[\frac{\partial \hat{u}}{\partial t}\right] + b(t, \xi)\hat{u} + \delta(t) \chi_{V_1}(\xi) (\hat{u}(+0, \xi) - \hat{u}(-0, \xi)) = \hat{f} + \delta(t)\chi_{V_1}(\xi) (\hat{u}(+0, \xi) - \hat{u}(-0, \xi)).$$

Taking an inverse Fourier transform (w. r. to ξ) of the equation (3.5), we have

(3.6) $$\frac{\partial u}{\partial t} + b(t, D_x)u = f - \delta(t) Kf(x),$$

where

$$\begin{aligned} Kf(x) &= -\mathcal{F}^{-1}(\chi_{V_1}(\xi) (\hat{u}(+0, \xi) - \hat{u}(-0, \xi))) \\ &= (2\pi)^{-n} \int_{s=-T}^T \int_{R_n} e^{ix\cdot\xi - B(0,\xi)+B(s,\xi)} \chi_{V_1}(\xi) \hat{f}(s, \xi) ds \ d\xi. \end{aligned}$$

Now

(3.7) $$D(H(t)Kf(x)) = \delta(t)Kf(x) + H(t)b(t, D_x)Kf(x).$$

From (3.6) and (3.7), we have

$$D(u + H(t)Kf(x)) = f + H(t)b(t, D_x)Kf(x).$$

We define a function $\nu(t, x)$ as follows:

$$\nu(t, x) = -(2\pi)^{-n} \int_{R_n} \int_{s=0}^t b(t, \xi) Kf(\xi) e^{ix.\xi - B(t,\xi)+B(s,\xi)} ds \ d\xi \quad \text{if } t \geqq 0$$

and $\nu(t, x) = 0$ if $t<0$.

Then we have

$$D(u+H(t)Kf(x)+\nu)=f,$$

and hence $w=u+H(t)Kf(x)+\nu$ is a solution of the equation (3.3). It is easy to show that w is C^1. The proof is complete.

Next we consider a operator $D=d_t+b(t, D_x)\Lambda$, where $b(t, \xi)=d_t B(t, \xi)$ and $x \in R^n$, in an open set $\Omega \in R^\nu$ containing the origin. If for some $\xi \in R_n \backslash 0$, $B(t, \xi)$ has a local minimum at an interior point of Ω, then we can not guarantee the solvability of D in dimension 0, in Ω. In this section we only concern the particular nonsolvable operators. Let $\Omega=(-T_1, T_1)\times\cdots\times(-T_\nu, T_\nu)$. We will make the assumptions for the nonsolvable operator D as follows:

There exists j $(1\leqq j\leqq\nu)$ such that both (3.8) and (3.9) holds;

(3.8) For some fixed $\xi \in R_n\backslash 0$, $B(t, \xi)$ has a local maximum at the origin, in which for any fixed $(t_1, \ldots, t_j, \ldots t_\nu)$ $B(t, \xi)$ is a decreasing function of t_j in $(0, T_j)$ and increasing function of t_j in $(-T_j, 0)$. Let $V=\{\xi \in R_n\backslash 0 : B(t, \xi)$ satisfies condition (3.8)$\}$. Then V is a cone since $B(t, \xi)$ is positive homogeneous of degree 1 with respect to ξ.

(3.9) For any fixed $\xi \in R_n\backslash V$, $B(t, \xi)$ is a monotone function of t_j in $(-T_j, T_j)$ or it has a local minimum at the origin, in which case $B(t, \xi)$ is an increasing function of t_j in $(0, T_j)$ and decreasing function of t_j in $(-T_j, 0)$.

PROPOSITION 3.2. *Let $B(t, \xi)$ satisfy the conditions* (3.8) *and* (3.9). *Let $f \in \mathscr{B}_D{}^1C_0{}^\infty(\Omega\times R^n)$ and*

$$Kf(t_1, \ldots, t_j, \ldots, t, x)=(2\pi)^{-n}\int_{s_j=-T_j}^{T_j}\int_{R_n} e^{ix\cdot\xi-B(t_1, \cdots, j_0, \cdots t_\nu)+B(t_1, \cdots, s_j, \cdots, t_\nu)}$$
$$\chi_V(\xi)\hat{f}(t_1, \cdots, s_j, \cdots, t_\nu, \xi)\, ds_j d\xi$$

be a real analytic function of x. Then

$$Du=f \text{ in } \Omega\times R^n,$$

has a C^1 solution.

Proof. cf. Proposition 3.1.

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