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UNIQUENESS IN THE CAUCHY PROBLEM FOR A CERTAIN FIRST ORDER LINEAR PARTIAL DIFFERENTIAL OPERATOR

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1. Introduction

In this article we prove uniqueness of the solution in the Cauchy problem

$$Lu=0$$
$$u(x,0)=0$$

where L is the first order linear partial differential operator

$$\frac{\partial}{\partial t}+ib(t)\frac{\partial}{\partial x}$$

and $b(t)$ is a strictly monotone real valued continuous odd function of t.

The proof is based on a technique, called as a *local constancy principle*, developed by Treves in [4] and [5] to construct a first order linear partial differential equation without any nonconstant solution. Thus the proof is quite different from the usual uniqueness proofs based on the Carleman estimate.

As $b(t)$ appearing in the definition of L can have $t=0$ as a zero point of infinite order, our result partially generalizes the uniqueness result of Strauss-Treves (cf. [3]) for the case where $t=0$ is a finite-order zero point of $b(t)$.

2. Theorems

Let Ω be an open neighborhood of the origin in R^2. We denote a point in R^2 by (x, t).

Let L be a linear partial differential operator of the first order defined by

$$L=\frac{\partial}{\partial t}+ib(t)\frac{\partial}{\partial x}. \tag{2.1}$$

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We assume that

(2. 2) $b(t)$ is real valued and continuous,

(2. 3) $b(t)=-b(-t)$ for any $t\in R$, and

(2. 4) $b(t)$ is strictly monotone.

Under these assumptions we have the following

THEOREM 1. *Let L be a linear partial differential operator of the first order given by* (2. 1)-(2. 4). *If u is a C^1 solution of $Lu=0$ in a neighborhood Ω of the origin, vanishing identically on $\Omega\cap\{(x,t)\mid t\leq 0\}$, then $u\equiv 0$ in a full neighborhood of the origin.*

Proof. Let U be an open neighborhood of origin invariant under the symmetry

$$(x,t)\rightarrow(x,-t).$$

We assume that U is contained in Ω.

Let S denote the intersection of U with the axist $t=0$, U^+ (resp., U^-) that with the half plane $t>0$ (resp., $t<0$). We denote by u^+ (resp., u^-) the restriction of u to U^+ (resp., U^-).

We note that the Cauchy problem

$$\frac{\partial z}{\partial t}+ib(t)\frac{\partial z}{\partial x}=0$$

$$z(x,0)=x$$

has a unique solution

$$z=x-iB(t)$$

where

$$B(t)=\int_0^t b(t)\,dt.$$

From the relations

$$z=x-iB(t),\quad \bar{z}=x+iB(t)$$

and the strict monotonicity of $b(t)$, we can solve (2. 9) with respect to x and t to get

$$x=x(z,\bar{z}),\quad t=t(z,\bar{z}).$$

It follows that x and t are C^1 functions of z and $\bar{z}$ if $\mathrm{Im}\ z\neq 0$ (i. e., $B(t)\neq 0$ or $t\neq 0$) and are continuous for all z and $\bar{z}$.

We set

$$h^+(z,\bar{z})=u^+(x,t),\ h^-(z,\bar{z})=u^-(x,t).$$

Then h^+ and h^- are two C^1 functions on

$$V=z(U^+)=z(U^-),$$

which can be continuously extended to $V\cup z(S)$. We notice that $z(S)$ is a nonempty open subset of the real axis Im $z=0$ and that $z(S)$ is a part of the boundary of V.

On the other hand,

$$\begin{aligned}0&=Lu^\pm\\&=(\partial h^\pm/\partial z)Lz+(\partial h^\pm/\partial\bar z)L\bar z\\&=(\partial h^\pm/\partial\bar z)L\bar z\end{aligned}$$

as $Lz=0$.

Now

$$L\bar z=\left(\frac{\partial}{\partial t}+ib(t)\frac{\partial}{\partial x}\right)(x+iB(t))=2ib(t).$$

Therefore, if $t\neq 0$ (as t is in $U^\pm$), we must have

$$\partial h^\pm/\partial\bar z=0.$$

In other words, h^+ and h^- are functions of z alone and $h^+(z)$ and $h^-(z)$ are holomorphic in V.

Since h^+ and h^- are equal on $z(S)$, they are equal in the whole V. In fact, h^+-h^- is holomorphic in V and is real valued on $z(S)$. Therefore, by the Schwarz reflection principle, h^+-h^- can be extended holomorphically across $z(S)$. Since $h^+-h^-=0$ on $z(S)$, $h^+-h^-\equiv 0$ in V by the connectedness of the latter.

Now the mapping

$$(x,t)\to z(x,t)$$

is a bijection of U^+ or of U^- onto V. Therefore we have

$$u^+(x,t)=u^-(x,-t)$$

for all $(x,t)\in U^+$.

But from the hypotheses

$$u^-(x,-t)=0 \text{ for all } (x,t)\in U^+.$$

Therefore

$$u^+(x,t)=0 \text{ for all } (x,t)\in U^+.$$

Thus $u\equiv 0$ on U, completing the proof.

THEOREM 2. *Let Ω be an open subset of R^2 and L a linear partial differential operator given by* (2.1)~(2.4). *Let Σ be a continuous curve in Ω separating $\Omega\backslash\Sigma$ into two parts. If u is a C^1 solution of $Lu=0$ in*

Ω and satisfies $u \equiv 0$ on one side of Σ, then $u \equiv 0$ in a full neighborhood of Σ.

Proof. If suffices to prove the theorem locally in a neighborhood U of an arbitrary point (x_0, t_0) in Σ.

If $t_0 \neq 0$, then under the local C^1 change of variables

$$y=x, \quad s=B(t)$$

in such a neighborhood, L becomes

$$L=b(t)\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial y}\right),$$

proportional to the Cauchy–Riemann operator and hence the unique continuation across Σ holds. In particular, $u \equiv 0$ in a full neighborhood of Σ.

Assume now $t=0$. Since L is invariant under x–translation, we may assume that $x_0=0$, that is, Σ passes through the origin. We may also asssume that U is invariant under the symmetry $(x, t) \rightarrow (x, -t)$.

Suppose the side of Σ on which $u \equiv 0$ intersects the half plane $t>0$. Let U^+ denote the intersection of U with that half plane $t>0$. As in the proof of the theorem 1, we have that u is a holomorphic function of $z=x-iB(t)$ in $z(U^+)$. Therefore $u \equiv 0$ in U^+.

But $u(x, -t)=u(x, t)$ if $(x, t) \in U$. Therefore $u \equiv 0$ in U^-, and hence $u \equiv 0$ on U. This completes the proof.

THEOREM 3. *Let Ω be a connected open subset of R^2 and L a linear partial differential operator of the first order given by* (2.1)~(2.4). *Let u be a C^1 solution of $Lu=0$ in Ω. If u vanishes on a nonempty open subset of Ω, then $u \equiv 0$ in Ω.*

Proof. Let $U=\{(x, t) \in R^2 | u(x, t)=0\}$ and U_0 be the connected component of U containing a nonempty open subset of Ω where $u \equiv 0$. Then U_0 is clearly closed in Ω since u is a C^1 function.

Let (x_0, t_0) be a limit point of U_0 in Ω. Then (x_0, t_0) is a boundary point of U_0 and lies on a C^1 curve Σ on one side of which u vanishes. By the previous theorem $u \equiv 0$ in a full neighborhood of (x_0, t_0). This shows that U_0 is also open.

Since Ω is connected, $\Omega=U_0$, completing the proof.

THEOREM 4. *Let Ω be a connected open subset of R^2. Let Σ be a C^1 curve separating $\Omega \backslash \Sigma$ into two parts. Let L be a first order linear partial differential operator in Ω satisfying* (2.1)~(2.4). *Then the Cauchy* *problem*

$$Lu=f(x,t)$$
$$u|_{\Sigma}=u_0(x)$$

where f and u_0 are continuous functions in Ω and Σ, respectively, has a unique C^1-solution.

Proof. As Σ is a C^1 curve, by a C^1 local coordinate change we may assume that Σ is the axis $t=0$.

Let u be a C^1 solution of

$$Lu=0$$
$$u(x,0)=0.$$

Then $H(t)u(x,t)$, where $H(t)$ is a Heaviside function, is a C^1 solution of $Lu=0$.

As $H(t)u(x,t)=0$ for $t<0$, by theorem 2 $H(t)u(x,t)\equiv 0$ in a full neighborhood of Σ. Similarly, $H(-t)u(x,t)\equiv 0$ and hence $u(x,t)\equiv 0$ in a full neighborhood of Σ.

Since Ω is connected, the same reasoning as in the proof of the theorem 3 completes the proof.

3. Remarks

1) The prototype of the first order linear partial differential operator satisfying the condition (2.1)~(2.4) is the generalized Mizohata type operator

$$M_k=\frac{\partial}{\partial t}+it^k\frac{\partial}{\partial x}$$

where k is an odd nonnegative integer.

2) We note that in the theorem 4 the C^1 curve Σ may be characteristic with respect to L.

3) In our discussions we may write L as

$$L=i(D_t+ib(t)D_x)$$

where $D_t=\frac{1}{i}\frac{\partial}{\partial t}$ and $D_x=\frac{1}{i}\frac{\partial}{\partial x}$. The (principal) symbol of $\frac{1}{i}L$ is $\tau+ib(t)\xi$. Thus $\tau+ib(t)\xi=0$ has a simple root $\tau=-ib(t)\xi$. If $b(t)$ is strictly increasing C^∞ function, then

$$\frac{\partial}{\partial t}(-b(t)\xi)\leq 0,$$

and hence $-b(t)\xi$ is strictly decreasing along the null bicharacteristic curve of τ in $T^*(\Omega)=\Omega\times R^2$. Therefore, in this particular case, our theorem 1 can be obtained as a consequence of §6, theorem 5′ in Nirenberg [2]. When $b(t)$ is strictly decreasing, however, theorem 1 does not follow from the Nirenberg's results.

4) We note that, in general, the first order linear partial differential operator L given by (2.1)~(2.4) does not satisfy the condition (P) of Nirenberg-Treves, and hence is not locally solvable. The function $b(t)$ in the definition of L is not (infinitely) oscillating in a neighborhood of origin, but still can have $t=0$ as a zero point of infinite order.

Therefore theorem 1 partially generalizes the following Strauss-Treves result (cf. [3], [6]).

THEOREM. *Let* $L=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}+C$ *be defined in a neighborhood of the origin in* R^2 *where* b *and* c *are* C^∞ *function. Suppose that* $t\to b(0,t)$ *has at* $t=0$ *a zero of finite order. Then there exists a neighborhood* V *of the origin in which every* C^1 *solution of* $Lu=0$ *satisfying* $u(x,t)=0$ *for* $t<0$ *vanishes.*

5) Consider $\frac{1}{i}L=D_t+ib(t)D_x$ with its symbol $p(x,t,\xi,\tau)=\tau+ib(t)\xi$.

At the point $x=t=0$, $\tau=0$, $\xi\neq 0$,

$$\text{Re } p=0,\ \text{Im } p=0$$

but

$$\{\text{Re } p,\ \text{Im } p\}=\frac{\partial b}{\partial t}\xi.$$

Therefore, if $b(t)$ is a C^∞ real valued function and $\frac{\partial b}{\partial t}\neq 0$ at $t=0$, then by the result of Alinhac [1] there exist C^∞ functions $u(x,t)$ in R^2 and $a(x,t)$ with support in $\{(x,t)\,|\,t\geq 0\}$ such that

$$Lu-au=0$$
$$\text{origin}\in\text{supp } u.$$

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