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ON AFFINE PARTS OF ALGEBRAIC THEORIES

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1. Introduction

The idempotent operations of an algebraic theory $\pi=(T, \eta, \mu)$ are the operations of a subtheory π' namely the *affine part* of π, [3]. A theory which coincides with its affine part is called *affine* [5], which is equivalent to saying: π is *affine* if and only if every singleton in a π-algebra is a subalgebra, or if and only if the free algebra generated by 1 is 1 [9]. For example, convex spaces, compact Hausdorff topological spaces or semilattices are models of affine theories.

This paper deals with affine parts of algebraic theories in a finitely complete category K, presented in a slightly different manner than in [6]. Our aim is to calculate the affine part of some algebraic theories (triples or monads) in K.

Let K be a finitely complete category and 1 a terminal object.

DEFINITION 1.1. An algebraic theory π in K is *affine* if $T(1)=1$.

For instance, if K is the category of abelian groups and Λ is a ring, then $\Lambda\otimes\{0\}=\{0\}$ means that Λ-modules is affine as an algebraic theory in K.

For any object A of K we denote by t_A the unique morphism from A to 1. Let $\pi=(T, \eta, \circ)$ be an algebraic theory in "clone" form [7] in K. Then the pullback

$$(1.2)\qquad \begin{array}{ccc} T'(A) & \xrightarrow{t_{T'}(A)} & 1 \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle i_A} \\ T(A) & \xrightarrow{T(t_A)} & T(1) \end{array}$$

defines the object $T'(A)$ and the monomorphism i_A. The naturality of

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η implies the existence and uniqueness of a morphism $\eta_A' : A \longrightarrow T'(A)$, such that $i_A \cdot \eta_A' = \eta_A$.

If $\alpha : A \longrightarrow T'(B)$ and $\beta : B \longrightarrow T'(C)$ are morphisms in the category K,

$$\begin{aligned} T(t_C) \cdot \{(i_C \cdot \beta) \circ (i_B \cdot \alpha)\} &= \{T(t_C) \cdot i_C \cdot \beta) \circ (i_B \cdot \alpha) \\ &= (\eta_1 \cdot t_{T,(C)} \cdot \beta) \circ (i_B \cdot \alpha) = (\eta_1 \cdot t_B) \circ (i_B \cdot \alpha) \\ &= T(t_B) \cdot i_B \cdot \alpha = \eta_1 \cdot t_{T'(B)} \cdot \alpha, \end{aligned}$$

thus there is a unique $\tau : A \longrightarrow T'(C)$, such that

$$i_C \cdot \tau = (i_C \cdot \beta) \circ (i_B \cdot \alpha).$$

From the well known fact that the embedding in K^K of the category of theories creates limits, and defining $\beta \circ' \alpha = \tau$ we obtain the subtheory π' of π given by i.

Also we obtain the multiplication of π', given as the unique morphism μ' such that $i_A \cdot \mu_A' = \mu_A \cdot i_A^2$.

To avoid confusion, we sometimes write $\eta^{T'}, i^T$ and $\mu^{T'}$.

If $\lambda : \boldsymbol{S} \longrightarrow \pi$ is a theory map, there exists a unique morphism of theories $\lambda' : \boldsymbol{S} \longrightarrow \pi'$ such that $\lambda \cdot i^S = i^T \cdot \lambda'$; i.e. the construction of the π' is a natural one. In fact, this construction is a coreflection of the category of theories.

DEFINITION 1.3. Given the algebraic theory π, the coreflection of π is called the *affine part* of π.

Note that π is affine if and only if $\pi \simeq \pi'$.

It follows from the definition 1.3 that the calculations of affine parts commute with all limits. In the next parapraph, we examine the affine parts of constructions that are not included in this case.

2. Properties of the Affine Parts

First we generalize a well known property of algebraic theories on the category *Set* of sets and functions (even for the infinitary ones), [3].

DEFINITION 2.1. Let $\boldsymbol{S}$ be a subtheory of π given by i. We say that $\boldsymbol{S}$ is *division-closed* if for any $\alpha : X \longrightarrow S(B)$, $v : A \longrightarrow S(B)$ and $w : X \longrightarrow T(A)$ with $(i_B \cdot v) \circ w = i_B \cdot \alpha$, there exists $\beta : X \longrightarrow S(A)$ with $w = i_A \cdot \beta$.

PROPOSITION 2.2. *The affine part of an algebraic theory π is division-closed.*

Proof. The existence of β is equivalent to saying that

$$T(t_A)\cdot w=\eta_1\cdot t_X$$

but this equality holds because

$$\begin{aligned}T(t_A)\cdot w&=(\eta_1\cdot t_A)\circ w=(\eta_1\cdot t_{T'(B)}\cdot v)\circ w\\&=(T(t_B)\cdot i_B\cdot v)\circ w=\{(\eta_1\cdot t_B)\circ i_B\cdot v)\circ w\\&=(\eta_1\cdot t_B)\circ(i_B\cdot\alpha)=\eta_1\cdot t_{T'(B)}\cdot\alpha=\eta_1\cdot t_X.\end{aligned}$$

It is easily proven that a subtheory of an affine theory is affine too. Then, from the exactness properties of the association $\pi\longrightarrow\pi'$, we obtain the following:

PROPOSITION 2.3. *Let $\boldsymbol{S}$ be a subtheory of π. If $\boldsymbol{S}'$ and π' are the respective affine parts, then for each object A in K we have that $S'(A)=S(A)\cap T'(A)$.*

We shall now be concerned with the calculus of the affine part of the composite of algebraic theories, definite by a distributive law [1].

Let $\pi=(T,\eta^T,\mu^T)$, $\boldsymbol{S}=(S,\eta^S,\mu^S)$ be algebraic theories on K. Let us assume that the composite theory exists; i. e. there exists a multiplication

$$m : TSTS\longrightarrow TS$$

with the following properties:

(i) $\pi\boldsymbol{S}=(TS,\eta^T\eta^S,m)$ is a theory in K.

(ii) The natural transformations $\eta^TS : S\longrightarrow TS$, and $T\eta^S : T\longrightarrow TS$ are theory maps.

(iii) The middley unitary law $m\cdot T\eta^S\eta^TS=TS$ holds.

PROPOSITION 2.4. *Let π and $\boldsymbol{S}$ be algebraic theories such that there exists the composite theory $\pi\boldsymbol{S}$. If T preserves pullbacks of type* (1.2), *then $(\pi\boldsymbol{S})'=\pi'\boldsymbol{S}'$.*

Proof. By the hypothesis on T, the composite pullback

$$\begin{array}{ccccc}T'S'(A)&\xrightarrow{i_{S'(A)}{}^T}&TS'(A)&\xrightarrow{T(i_A{}^S)}&TS(A)\\ {\scriptstyle t_{T'S'(A)}}\downarrow&&\downarrow{\scriptstyle T(t_{S'(A)})}&&\downarrow{\scriptstyle TS(t_A)}\\ 1&\xrightarrow{\eta_1{}^T}&T(1)&\xrightarrow{T(\eta_1{}^S)}&TS(1)\end{array}$$

allows us to assert that $(TS)'=T'S'$.

Now

$$\begin{aligned}(i^Ti^S)_A\cdot(\eta^{T'}\eta^{S'})_A&=T(i_A{}^S)\cdot i_{S'(A)}{}^T\cdot\eta_{S'(A)}{}^{T'}\cdot\eta_A{}^{S'}\\&=T(i_A{}^S)\cdot\eta_{S'(A)}{}^T\cdot\eta_A{}^{S'}=(\eta^T\eta^S)_A,\end{aligned}$$

hence, $\eta^{T'}\eta^{S'}$ is the unit of the affine part of $\pi\boldsymbol{S}$, which will be the theory $(\pi S)'=(T'S', \eta^{T'}\eta^{S'}, m')$, m' being determined by

$$m_A \cdot (i^T i^S)^2{}_A=(i^T i^S)_A \cdot m_A'.$$

On the other hand, since for each object A of K

$$T(i_A{}^S) \cdot i_{S'(A)}{}^T \cdot \eta_{S'(A)}{}^T=\eta_{S(A)}{}^T \cdot i_A{}^S,$$

holds, it follows that $\eta^{T'}S' : S' \longrightarrow \pi'S'$ is a theory map.

Similarly, $T'\eta^{S'} : \pi' \longrightarrow \pi'S'$ is also a theory map.

It remains to prove the middley unitary law.

By definition of η', we obtain:

$$(i^T i^S)_A \cdot m_A' \cdot (T'\eta^{S'}\eta^{T'}S')_A=(i^T i^S)_A \cdot id_{T'S'(A)},$$

$$t_{T'S'(A)} \cdot m_A' \cdot (T'\eta^{S'}\eta^{T'}S')_A=t_{T'S'(A)} \cdot id_{T'S'(A)}.$$

Therefore, $m_A'.(T'\eta^{S'}\eta^{T'}S')_A=id_{T'S'(A)}$ as was to be shown.

If $l=m \cdot \eta^T S T\eta^S : S\pi \longrightarrow \pi S$ is the distributive law of S over π, we obtain the following relation:

$$i^T i^S \cdot l'=i^T i^S \cdot m' \cdot \eta^{T'}S' T'\eta^{S'}=m \cdot \eta^T S T\eta^S \cdot i^S i^T=l \cdot i^S i^T,$$

l' being the correspondent distributive law between the affine parts.

3. Examples and Applications

3.1. Let L be a set. The adjunction

$$Set^\circ \underset{(-,L)^\circ}{\overset{(-,L)}{\rightleftarrows}} Set$$

where $(-,L)$ represents the functor $Hom_{Set}(-,L)$, gives rise to an algebraic theory $\boldsymbol{D}$ in *Set*, the "fuzzy" *Double Dualization Theory*. The description of $\boldsymbol{D}=(D,\eta,\mu)$ is:

$$D(X)=((X,L),L)$$

$$Df(\Psi)=\Psi \cdot (f,L) \text{ for each } f\in(X,Y) \text{ and } \Psi\in((X,L),L)$$

$$(\eta_X(x))(A)=A(x) \text{ for each } x\in X \text{ and } A\in(X,L)$$

$$(\mu_X(\phi))(A)=\phi(\eta_{(X,L)}(A)) \text{ for each } A\in(X,L) \text{ and } \phi\in D^2(X).$$

If we consider the pullback diagram

$$\begin{array}{ccc} D'X & \xrightarrow{t_{D'(X)}} & 1 \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle \eta_1} \\ DX & \xrightarrow{D(t_X)} & D1 \end{array}$$

$\Psi\in D'(X)$ if and only if $D(t_X)(\Psi)=\eta_1(1)=id_L$. But, $D(t_X)(\Psi)=$

$\Psi \cdot (t_X, L)$. Hence

(*) $\Psi \in D'(X) \iff \Psi(k_l)=l$ for each $l\in L$

being $k_l : X \longrightarrow L$ given by $k_l(x)=l$ for each $x\in X$.

Now let us assume that $L=2$. Then $\boldsymbol{D}$ is the *Double Power-Set Theory* $D(X)=P(P(X))$, i.e. the set of all collections of subsets of X and $Set^{\boldsymbol{D}}$ is the category of Complete Atomic Boolean Algebras. In this case (*) becomes

$$\Psi \in D'X \iff X\in \Psi \text{ and } \Phi \notin \Psi.$$

This caracterization can also be obtained from a syntactic point of view, taking into account that each $\Psi\in PPX$ has a unique representation in the variables $\eta_X(x)$

$$\Psi = \bigcup_{A\in\psi} [(\bigcap_{x\in A} p(x)) \cap (\bigcap_{x\notin A} (p(x))')]$$

$p(x)=\eta_X(x)$ being the principal ultrafilter on x.

Finally note that, as a consequence, for a set X with finite cardinal n, there exist exactly 2^{2n-2} collections Ψ in $D'(X)$.

3.2. We may now show how Proposition 2.3 allows us to identify the affine parts of some subtheories of $\boldsymbol{D}$.

i) Let $D_{\leq}(A)=\{\Psi\subset P(A)/\Psi$ is order filter preserving$\}$ be the algebraic theory whose models are the completely distributive complete lattices.

$$D'_{\leq}(A)=\{\Psi\in D_{\leq}(A)/\Phi\notin\Psi\}.$$

ii) $Q(A)=\{F/F$ is quasifilter on $A\}$, defines an algeraic theory. The correspondent category of models has objects complete lattices satisfying

$$Inf(Sup A_i/i\in I)=Sup(Inf(a_i/i\in I)/(a_i)\in \prod_{i\in I} A_i),$$

for each family $(A_i)_{i\in I}$ of directed subsets, and whose morphisms preserve all infima and all suprema of directed subsets. Then, $Q'(A)=\{F/F$ is a filter on $A\}$.

iii) We denote by P the covariant powerset functor on sets. Singletons and set unions define natural transformations η and μ. This data define the well known algebraic theory $\boldsymbol{P}=(P,\eta,\mu)$ called the *Powerset Theory*. $Set^{\boldsymbol{P}}$ may be identified with the category of complete semilattices. $\boldsymbol{P}$ is a subtheory of $\boldsymbol{D}$ in this way $\lambda_A : PA \longrightarrow DA$,

$\lambda_A(X)=\{S/X\subset S\}$. Then,

$$P'(A)=\{X\subset A/\{S/X\subset S\} \text{ is a filter}\}$$
$$=\{X\subset A/X\neq\Phi\}.$$

3.3. Finally, we list here some examples where the Proposition 2.4 can be applied.

i) Let E be a set. Let us interpret E as an algebraic theory $\boldsymbol{E}$ in *Set* whose models form the slice category E/Set. This theory admits a distributive law over any algebraic theory π. Hence the composite theory $\pi\boldsymbol{E}$ exists.

For example, for $\pi=\boldsymbol{P}$, $(PE)'(A)=\{S\subset A/S\neq\Phi\}$.

ii) Let $\boldsymbol{S}$ be the free monoid theory in *Set*, and π the free abelian group theory. As is well known, there exists a distributive law of $\boldsymbol{S}$ over π. The composite $\pi\boldsymbol{S}$ is the free ring theory and $TS(X)$ is the polynomial ring $\boldsymbol{Z}[X]$ with the elements of X as noncommuting indeterminates. Then, as $\boldsymbol{S}'$ is the identity, we obtain $(\pi\boldsymbol{S})'=\pi'$.

iii) A monoid M can be interpreted as an aigebraic theory in *Set* via cartesian product in the obvious way. Let Set^M be the corresponding algebraic category of M-sets. If π is any theory, there is the composite theory $\pi\boldsymbol{M}$ whose algebras are the π-algebras equipped with M-operations and the elements of M act as π-homomorphisms. If π is in the hypothesis of 2.4, then $(\pi\boldsymbol{M})'=\pi'$ because the theory $\boldsymbol{M}'$ is isomorphic to the identity.

iv) In [2], Bunge shows that if a theory $\boldsymbol{S}$ preserves all powers, then there exists the composite $\pi\boldsymbol{S}$ for any theory π. In particular, we can take $\boldsymbol{S}$ as the n-power theory given through the following data: the n-power functor $(-)^n : Set \longrightarrow Set$, $\eta_A : A \longrightarrow A^n$ is the exponencial adjoint of the projection $n\times A \longrightarrow A$, and $\mu_A : (A^n)^n \longrightarrow A^n$ is induced by the diagonal map $n \longrightarrow n\times n$.

We thus have $(T(-)^n)'(A)=T'(A^n)$ for any theory π in the hypothesis of 2.4.

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