J. Korean Math. Soc. 23(1986), No. 2, pp. 151-157

LOCALIZATIONS AND GENERALIZED GOTTLIEB SUBGROUPS

Moo Ha Woo and Jae-Ryong Kim

This work is a continuation of the study of evaluation subgroups defined by Gottlieb [2] and Varadarajan [8] among others. Gottlieb studied the evaluation subgroups $G(X)$ of $\pi_1(X)$ and $G_n(X)$ of $\pi_n(X)$ extensively. Varadarajan generalized $G_n(X)$ to $G(A,X)$ and dualized. On the other hand, Lang, Jr. [5] proved that $G_n(X)_p \cong G_n(X_p)$, where $X_p, G_n(X)_p$ are the localizations at prime p of the connected simple finite CW-complex X and the evaluation subgroup $G_n(X)$ respectively. The purpose of this paper is to prove that $G(A,X)_p \cong G(A,X_p)$ for some suitable spaces A and X.

1. Introduction

General references for the localization theory are [1], [3], [4] and [7]. We review some of these results here.

If P is a collection of primes, we denote by P' the complementary collection of primes. If the integer n is a product of primes in P', we write $n \in P'$.

A group G is said to be P-local if $x \longrightarrow x^n, x \in G$ is bijective for all $n \in P'$. In the category H of all nilpotent groups, a homomorphism $e: G \longrightarrow G_p$ is said to be P-localizing map if G_p is P-local and if e^* : $\mathrm{Hom}(G_p, K) \cong \mathrm{Hom}(G, K)$, provided $K \in H$ with K P-local.

We recall that a pointed space X is said to be *nilpotent* if it is of the pointed homotopy type of a connected CW-complex, and if moreover $\pi_1(X)$ is nilpotent and operates nilpotently on the higher homotopy groups of X. If X is nilpotent and A is a compact polyhedron, then every component of the function space X^A with compact-open-topology is nilpotent [3].

Received February 15, 1986.
This research is supported by a grant from the Korea Science and Engineering Foundation Grant.

Let NH be the homotopy category of nilpotent CW-complexes. Plainly, the simple CW-complexes are in NH; in paticular, NH contains all connected Hopf spaces.

Assume that the spaces A and X are nilpotent finite CW-complexes. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$.

We say that a nilpotent space X is a *p-local* if all its homotopy groups are p-local. A map $e_p: X \longrightarrow Y$ *p-localizes* X if Y is p-local and there is natural 1-1 correspondence $e_p^*: [Y, Z] \cong [X, Z]$ for all p-local Z. It is equivalent that $e_{p*}: \pi_n(X) \longrightarrow \pi_n(Y)$ p-localizes $\pi_n(X)$, $n \geqq 1$. That is, e_{p*} is a p-isomorphism (p-injective and p-surjective) and $\pi_n(Y)$ is p-local. Every X in NH admits a p-localization.

We say that a homomorphism $\phi: A \longrightarrow B$ from a group A into a group B is *p-injective* if the kernel of ϕ consists of elements of finite order prime to p; and that it is *p-surjective* if, given any $b \in B$, there exists a positive integer n prime to p such that $b^n \in \mathrm{Im}\phi$.

If we let $\hat{e}_p: (X^A, f) \longrightarrow (X_p^A, e_p f)$ be defined by $\hat{e}_p(g) = e_p g$, we then have

PROPOSITION 1.1. *If A is a connected finite CW-complex and X is nilpotent, then $\hat{e}_p$ p-localizes* [3], [4].

PROPOSITION 1.2. *Let A be a finite connected CW-complex and X be a connected H-space. Then the localization map $e_p: X \longrightarrow X_p$ induces a homomorphism*

$$e_{p\#}: [A, X] \longrightarrow [A, X_p]$$

of monoids. Moreover $e_{p\#}$ is p-bijective [4].

Let A be a connected finite CW-complex and X be a connected H-space with finitely generated homotopy groups in each dimension. Then the p-localization of $[A, X]$ is isomorphic to $[A, X_p]$ for every prime p [7]. Since $(\Omega X)_p \equiv \Omega(X_p)$ for X in NH, we have

PROPOSITION 1.3. *Let A be a connected finite CW-complex with suspension structure and X be a nilpotent space with finitely generated homotopy groups in each dimension. Then for every prime p, the p-localization of $[A, X]$ is isomorphic to $[A, X_p]$.*

2. Generalized Gottlieb Subgroups

For simplicity, we sometimes use the same symbol for a map and its homotopy class. X^X shall denote the space of free maps from X to X with 1_X as base point. The evaluation map $\omega : X^X \longrightarrow X$ is defined to be $\omega(f)=f(*)$ for each $f \in X^X$. The folding map $\nabla : X \vee X \longrightarrow X$ is given by $\nabla(x, *)=\nabla(*, x)=x$ for each $x \in X$. Frequently $1, 1_X$ and 1_{X_p} will be reserved for the identity maps and j for the inclusion map $j : X \vee A \longrightarrow X \times A$ respectively.

DEFINITION 2.1. A map $f : A \longrightarrow X$ is said to be *cyclic* if there exists a map $F : X \times A \longrightarrow X$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} X \times A & \xrightarrow{F} & X \\ \big\downarrow j & \nearrow \nabla(1 \vee f) & \\ X \vee A & & \end{array}$$

that is, $Fj \sim \nabla(1 \vee f)$. Since j is a cofibration, this is equivalent to saying that we can find a map $G : X \times A \longrightarrow X$ such that $Gj=\nabla(1 \vee f)$, that is, $G|_X=G|_{X\times *}=1_X$ and $G|_A=G|_{*\times A}=f$. We call such a map G *an associated map* of f. The set of all homotopy classes of cyclic maps from A to X is denoted by $G(A, X)$ and is called *the Gottlieb subset* of $[A, X]$. Now we recall the followings:

PROPOSITION 2.1. *Let X be a locally compact Hausdorff space, Z a Hausdorff space and Y any space. Then the function spaces $(Y^X)^Z$ and $Y^{X\times Z}$ are homeomorphic and a homeomorphism $H : (Y^X)^Z \equiv Y^{X\times Z}$ is given by $H(g)(x, z)=g(z)(x)$ for each $g : Z \longrightarrow Y^X$, $x \in X$, $z \in Z$. Furthermore, $f \sim g$ iff $H(f) \sim H(g)$.*

PROPOSITION 2.2. *Let X be a space having the homotopy type of a locally finite CW-complex and A any Hausdorff space. Suppose $\omega : X^X \longrightarrow X$ is the evaluation map where X^X is the space of free maps from X to X with 1_X as base point. Then $\omega_\#([A, X^X])=G(A, X)$ as set, where $\omega_\#$ is the induced function of ω* [6].

Under the same hypothesis as the above theorem, if in addition, A is a suspension, then $\omega_\#([A, X^X])=G(A, X)$ as group. This justifies the term evaluation subgroup.

THEOREM 2.3. *Let X and A be nilpotent, finite CW-complexes and $e_p : X \longrightarrow X_p$ be the p-localization map. Let $X_p{}^{X_p}$ and $X_p{}^X$ be the spaces of free maps from X_p to X_p and X to X_p with 1_{X_p} and e_p as base points respectively. Then there is* 1–1 *correspondence*

$$[A, X_p{}^{X_p}] \overset{\theta}{\Longleftrightarrow} [A, X_p{}^X].$$

Proof. If $f \in [A, X_p{}^{X_p}]$, by Proposition 2.1, we have

$$H(f) : X_p \times A \longrightarrow X_p \text{ such that } H(f)|_{X_p} = 1_{X_p}.$$

Now define $\phi(H(f)) : X \times A \longrightarrow X_p$ by the composition

$$X \times A \xrightarrow{e_p \times 1} X_p \times A \xrightarrow{H(f)} X_p.$$

Then $\phi(H(f))|_X = e_p$.

Let $H' : (X_p{}^X)^A \equiv X_p{}^{X \times A}$ be the homeomorphism given by $H'(g)(x, a) = (g(a))(x)$.

Now define $\theta : [A, X_p{}^{X_p}] \longrightarrow [A, X_p{}^X]$ by $\theta(f) = H'^{-1}(\phi(H(f)))$.

Conversely if $g \in [A, X_p{}^X]$, then we have

$$H'(g) : X \times A \longrightarrow X_p \text{ such that } H'(g)|_X = e_p.$$

Consider the following commutative diagram obtained by localization[1]:

$$\begin{array}{ccc} X \times A & \xrightarrow{H'(g)} & X_p \\ \downarrow{\scriptstyle e_p \times 1} & \searrow{\scriptstyle e_p \times e_p'} & \uparrow{\scriptstyle H'(g)_p} \\ X_p \times A & \xrightarrow[1 \times e_p']{} & X_p \times A_p \end{array}$$

where $e_p' : A \longrightarrow A_p$ is the p-localization map.

Define $\psi(H'(g)) : X_p \times A \longrightarrow X_p$ by the composition

$$X_p \times A \xrightarrow{1 \times e_p'} X_p \times A_p \xrightarrow{H'(g)_p} X_p.$$

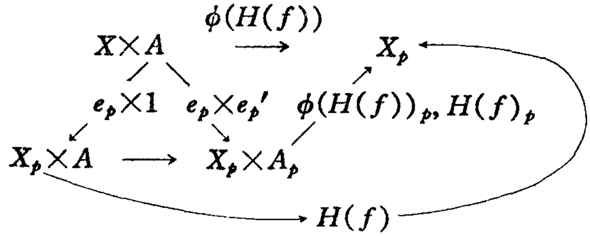
Using the universality of the localization, we have

$$\psi(H'(g))|_{X_p} = 1_{X_p}.$$

Thus we can define $\theta^{-1} : [A, X_p{}^X] \rightarrow [A, X_p{}^{X_p}]$ as

$$\theta^{-1}(g) = H^{-1}(\psi(H'(g))).$$

Using the following commutative diagram defined by localization and its universality, we can have $\psi\phi(H(f)) = H(f)$:



Thus we have $\theta^{-1}\theta(f)=H^{-1}\psi H'H'^{-1}\phi H(f)=f$. Similarly we can have $\theta\theta^{-1}(g)=g$. This completes the proof.

3. Main Theorems

Let p be a prime. Let X and A be nilpotent, finite CW-complexes and $e_p : X \longrightarrow X_p$ be the p-localization map. Moreover let A have a suspension structure. Then Proposition 1. 3 tells us that e_p induces the p-localizing homomorphism

$$e_{p\#} : [A, X] \longrightarrow [A, X_p], e_{p\#}(f)=e_p f.$$

THEOREM 3. 1. *If* $f\in G(A, X)$, *then* $e_{p\#}(f)\in G(A, X_p)$.

Proof. Since $f\in G(A, X)$, there exists an associated map

$$G : X\times A \longrightarrow X \text{ such that } G|_A=f, G|_X=1_X.$$

Let $F=e_pG : X\times A \longrightarrow X_p$.
By Theorem 2. 3, we have

$$\theta^{-1}(F) : X_p\times A \longrightarrow X_p \text{ such that } \theta^{-1}(F)|_A=e_p f.$$

Since $\theta^{-1}(F)|_{X_p}=1_{X_p}$, $e_{p\#}(f)=e_p f\in G(A, X_p)$.

THEOREM 3. 2. *The p-localization of the generalized Gottlieb subgroup $G(A, X)$ is isomorphic to the generalized Gottlieb subgroup $G(A, X_p)$.*

Proof. It suffices to show that the homomorphism

$$e_{p\#} : G(A, X) \longrightarrow G(A, X_p)$$

is p-bijective and $G(A, X_p)$ is p-local. Since $e_{p\#}:[A, X] \longrightarrow [A, X_p]$ is p-injective, $e_{p\#}|_{G(A,X)}$ is clearly p-injective. If $g\in G(A, X_p)$, by Theorem 2. 2, there is $\tilde{g}\in[A, X_p{}^{X_p}]$ such that

$$\omega_\#(\tilde{g})=g.$$

Consider the following commutative diagrams

$$\begin{array}{ccc} [A, X^X] & \xrightarrow{\omega_\#} & [A, X] \\ \downarrow \hat{e}_{p\#} & & \downarrow e_{p\#} \\ [A, X_p{}^X] & \xrightarrow{\omega_\#} & [A, X_p] \\ \cong \downarrow \theta & \nearrow \omega_\# & \\ [A, X_p{}^{X_p}] & & \end{array}$$

where $\hat{e}_{p\#}$ induced by $\hat{e}_p$ defined in Proposition 1. 1 and $\omega_\#$ induced by evaluation maps. Since X^X is clearly an H-space, $\hat{e}_{p\#}$ is p-injective by Proposition 1. 2. Thus there exist an $x \in [A, X^X]$ and integer q relatively prime to p such that

$$\hat{e}_{p\#}(x) = \theta(\tilde{g})^q.$$

Hence $e_{p\#}\omega_\#(x) = \omega_\#\hat{e}_{p\#}(x) = \omega_\#(\theta(\tilde{g})^q) = \omega_\#(\tilde{g})^q = g^q$.

Moreover $[A, X_p{}^X]$ is p-local, so that we can easily prove that $G(A, X_p)$ is p-local.

THEOREM 3. 3. *The generalized Gottlieb subgroup* $G(A, X_p)$ *is isomorphic to the generalized Gottlieb subgroup* $G(A_p, X_p)$.

Proof. Since $e_p{}^* : [A_p, X_p] \longrightarrow [A, X_p]$, $e_p{}^*(f) = fe_p$ is an isomorphism, it suffices to show that

$$e_p{}^* : G(A_p, X_p) \longrightarrow G(A, X_p)$$

is an epimorphism.

Let $f \in G(A, X_p)$. Then there is

$$F : X_p \times A \longrightarrow X_p \text{ such that } F|_A = f, F|_{X_p} = 1_{X_p}.$$

Consequently we have

$$F_p : X_p \times A_p = (X_p \times A)_p \longrightarrow X_p \text{ such that } F_p|_{A_p} = f_p, F_p|_{X_p} = 1_{X_p}.$$

This means that $f_p \in G(A_p, X_p)$. Moreover $e_p{}^*(f_p) = f_p e_p = f$. This completes the proof.

References

1. M. Arkowitz, *Localization and H-spaces*, AARHUS Universitet Lecture Notes Series **44**(1976).
2. D.H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math.

91(1969) 729–756.
3. P.J. Hilton, *Localization in nilpotent spaces*, Springer Lect. Notes **428**(1974) 18–43.
4. ———, Guido Mislin, and Joe Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland Math. Studies **15**.
5. G.E. Lang, Jr., *Localization and Evaluation subgroups*, Proc. Amer. Math. Soc. **50**(1975) 489–494.
6. K.L. Lim, *On Evaluation subgroups of generalized homotopy groups*, Can. Math. Bull. **27**(1984) 78–86.
7. Albert O. Shar, *Localization, Algebraic Loops and H-spaces I, II*, Can. J. Math. **31**(1979) 427–435, 812–817.
8. K. Varadarajan, *Generalized Gottlieb groups*, J. Indian Math. Soc. **33**(1969) 141–164.

Korea University
Seoul 132, Korea
and
Kookmin University
Seoul 132, Korea