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LIFTS OF DERIVATIONS TO THE TANGENT BUNDLE OF P^r-VELOCITIES

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Introduction

Let M be an n-dimensional C^∞ manifold and $T_p{}^rM$ the tangent bundle of p^r-velocities of M. In this paper, the λ-lift to $T_p{}^rM$ of derivations of the tensorial algebra on M is defined and their properties are established. The results obtained generalize those of K. Yano and S. Ishihara for the tangent bundle TM of M [6], C. Yuen for the tangent bundle of order 2, T^2M, of M [7] and ourselves for the frame bundle FM of M [4].

1. The tangent bundle of p^r-velocities

Let M be n-dimensional manifold. We denote by $T_p{}^rM$ the set of all r-jets at 0 of differentiable mappings of open neighborhoods of 0 in $\boldsymbol{R}^p$ onto open subsets of M. Let $\pi : T_p{}^rM \longrightarrow M$ be the target projection $\pi\ (j_0{}^r\varphi) = \varphi(0)$. Then, $\pi : T_p{}^rM \longrightarrow M$ has a natural bundle structure over M. $T_p{}^rM$ is called the tangent bundle of p^r-velocities of M [5]. Let us observe that $T_1{}^1M = TM$ is nothing but the tangent bundle of M and $T_1{}^rM = T^rM$ is the tangent bundle of order r of M.

Let $N(r, p)$ denote the set of all p-tuples $\nu = (\nu_1, \ldots, \nu_p)$ of non-negative integers such that $|\nu| = \nu_1 + \cdots + \nu_p \leq r$. Every chart (U, x^i) on M induces a chart

$$\{\pi^{-1}U = T_p{}^rU, x_i{}^{(\nu)},\ \ \nu \in N(p, r)\}$$

on $T_p{}^rM$, called the induced chart, where

$$x_i{}^{(\nu)}(j_0{}^r\varphi) = \frac{1}{\nu!} D_\nu(x^i \cdot \varphi)\ (0)$$

If f is a differentiable function on M and $\nu \in N(p, r)$, then we define the ν-*lift* of f as the function $f^{(\nu)}$ on $T_p{}^rM$ given by

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$$f^{(\nu)}(j_0{}^r\varphi)=\frac{1}{\nu!}D_\nu(f\circ\varphi)\quad(0)$$

It is convenient to define $f^{(\nu)}=0$ if $\nu\notin N(p,r)$. It is easy to verify that

$$(af+bg)^{(\nu)}=af^{(\nu)}+bg^{(\nu)}$$
$$(fg)^{(\nu)}=\sum_{\mu\in N(p,r)}f^{(\mu)}g^{(\nu-\mu)}$$

for all functions f, g and all real numbers a, b. Vector fields on $T_p{}^rM$ are characterized by their actions on functions of type $f^{(\nu)}$. More precisely, we have

PROPOSITION 1.1. *Let X, Y be vector fields on $T_p{}^rM$ such that $\tilde{X}f^{(\nu)}=\tilde{Y}f^{(\nu)}$, for every function f on M and all $\nu\in N(p,r)$. Then $\tilde{X}=\tilde{Y}$.*

The proof is a straighforward verification and can be found in [5]. Moreover, A. Morimoto has proved the following proposition

PROPOSITION 1.2. *If X is a vector field on M, then for every $\lambda\in N(p,r)$ there exists one and only one vector field $X^{(\lambda)}$ on $T_p{}^rM$ such that*

$$X^{(\lambda)}f^{(\nu)}=(Xf)^{(\nu-\lambda)}$$

for any function f on M and $\nu\in N(p,r)$.

$X^{(\lambda)}$ is called the *λ-lift* of X to $T_p{}^rM$. It is convenient to define $X^{(\lambda)}=0$ if $\lambda\notin N(p,r)$.

One can easily verify that

$$[X^{(\lambda)}, Y^{(\mu)}]=[X, Y]^{(\lambda+\mu)}\tag{1.1}$$

for any vector fields X, Y on M and $\lambda, \mu\in N(p,r)$.

By a similar device of those used in the Proposition 1.1, we have

PROPOSITION 1.3. *Let $\tilde{F}, \tilde{G}$ be tensor fields of type $(1,s)$ $s>0$, on $T_p{}^rM$, such that*

$$\tilde{F}(X_1{}^{(\lambda_1)},\dots,X_s{}^{(\lambda_s)})=\tilde{G}(X_1{}^{(\lambda_1)},\dots,X_s{}^{(\lambda_s)})$$

for any arbitrary vector fields $X_1,\dots,X_s$ on M, $\lambda_1,\dots,\lambda_s\in N(p,r)$. Then $\tilde{F}=\tilde{G}$.

A. Morimoto has proved the following proposition [5]

PROPOSITION 1.4. *Let F be a tensor field of type $(1,s)$, $s>0$, on M. Then, for every $\lambda\in N(p,r)$, there exists one and only one tensor field $F^{(\lambda)}$ of type $(1,s)$ on $T_p{}^rM$ such that*

$$F^{(\lambda)}(X_1{}^{(\mu_1)},\dots,X_s{}^{(\mu_s)})=(F(X_1,\dots,X_s))^{(\lambda+\mu)}$$

for any vector fields $X_1, \ldots, X_s$ *on* M, *and* $\mu_1, \ldots, \mu_s \in N(p, r)$, *where* $\mu = \mu_1 + \cdots + \mu_s$.

$F^{(\lambda)}$ is called the *λ-lift* of F to $T_p{}^rM$. As above, it is convenient to define $F^{(\lambda)} = 0$, if $\lambda \notin N(p, r)$. If $\lambda = (0, \ldots, 0)$, then the λ-lift $X^{(\lambda)}$ (resp., $F^{(\lambda)}$) to $T_p{}^rM$ of a vector field X (resp., a tensor field F of type $(1, s)$ on M, will be called the *complete lift* to $T_p{}^rM$ of X (resp., F) and denoted by X^C (resp., F^C).

Now, we consider a linear connection ∇ on M. In [5], A. Morimoto has proved the following result

PROPOSITION 1.5. *There exists one and only one linear connection* ∇^C *on* $T_p{}^rM$ *defined by the following condition*

$$\nabla^C{}_{X^{(\lambda)}} Y^{(\mu)} = (\nabla_X Y)^{(\lambda+\mu)},$$

for any vector field X *on* M *and* $\lambda, \mu \in N(p, r)$.

The connection ∇^C in the Proposition 1.5 is called the *complete lift* of ∇ to $T_p{}^rM$.

We remark that the r-frame bundle F^rM of M is an open and dense subset of the tangent bundle $T_n{}^rM$ of n^r-velocities. Then, we can consider the restriction to F^rM of the λ-lifts $f^{(\lambda)}, X^{(\lambda)}, F^{(\lambda)}$ defined above for $T_n{}^rM$, which will be called and denoted in the same manner. J. Gancarzewicz [2] has proved the following proposition

PROPOSITION 1.6. *Let* $\tilde{F}, \tilde{G}$ *be tensor fields of type* $(1, s)$, $s > 0$, *on* F^rM *such that*

$$\tilde{F}(X_1{}^C, \ldots, X_s{}^C) = \tilde{G}(X_1{}^C, \ldots, X_s{}^C),$$

$X_1, \ldots, X_s$ *vector fields on* M. *Then,* $\tilde{F} = \tilde{G}$.

2. Lifts of derivations to $T_p{}^rM$

Let $\mathcal{T}(M) = \sum \mathcal{T}_s{}^r(M)$ be the tensorial algebra of the tensor fields on M. By a derivation of $\mathcal{T}(M)$, we shall mean a mapping $D : \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$ which satisfies the following conditions:

(a) $D : \mathcal{T}_s{}^r(M) \longrightarrow \mathcal{T}_s{}^r(M)$
(b) $D(S+T) = DS + DT$, $S, T \in \mathcal{T}_s{}^r(M)$
(c) $D(S \otimes T) = (DS) \otimes T + S \otimes (DT)$, $S, T \in \mathcal{T}(M)$
(d) D commutes with every contraction of a tensor field

The set $\mathcal{D}(M)$ of all derivations of $\mathcal{T}(M)$ forms a Lie algebra over

$\boldsymbol{R}$ of an infinite dimension with respect to the natural addition and multiplication and the bracket operation defined by $[D, D']K = D(D'K) - D'(DK)$. Two derivations D and D' of $\mathcal{T}(M)$ coincide if and only if they coincide on $\mathcal{T}_0^0(M)$ and $\mathcal{T}_0^1(M)$, i. e., on the functions and the vector fields on M. Every derivation D of $\mathcal{T}(M)$ can be decomposed uniquely as follows

$$D = \mathcal{L}_X + i_F,$$

where $\mathcal{L}_X$ is the Lie derivative with respect to a vector field X and i_F is the derivation defined by a tensor field F of type $(1, 1)$ on M. The set $\mathcal{L}(M)$ of Lie derivatives $\mathcal{L}_X$ forms a subalgebra of the Lie algebra $\mathcal{D}(M)$. On the other hand, the set $\mathcal{E}(M)$ of all derivations i_F is an ideal of the Lie algebra $\mathcal{D}(M)$.

PROPOSITION 2.1. *Two derivations D and D' of $\mathcal{T}(T_p^r M)$ coincide if and only if*

(a) $Df^{(\lambda)} = D'f^{(\lambda)}$, *for any function f on M and $\lambda \in N(p, r)$*

(b) $DY^{(\lambda)} = D'Y^{(\lambda)}$, *for any vector field Y on M and $\lambda \in N(p, r)$.*

Proof. It is sufficient to show that if $Df^{(\lambda)} = 0, DY^{(\lambda)} = 0$, for any function f and any vector field Y on M, $\lambda \in N(p, r)$, then $D = 0$. If $D = \mathcal{L}_{\tilde{X}} + i_{\tilde{F}}$, then

$$Df^{(\lambda)} = \mathcal{L}_{\tilde{X}} f^{(\lambda)} = \tilde{X} f^{(\lambda)} = 0,$$

on M and $\lambda \in N(p, r)$. Taking into account Proposition 1.1, we deduce $\tilde{X} = 0$. Thus, $D = i_{\tilde{F}}$ and hence

$$DY^{(\lambda)} = i_{\tilde{F}} Y^{(\lambda)} = \tilde{F} Y^{(\lambda)} = 0$$

for any vector field Y on M and $\lambda \in N(p, r)$. Then, from Proposition 1.3, we deduce $\tilde{F} = 0$.

REMARK. If we consider the case of $F^r M$, the part (b) of the proposition 2.1 can be established as follows

(b)′ $DY^C = D'Y^C$, for every vector field Y on M.

Let $D = \mathcal{L}_X + i_F$ be a derivation of $\mathcal{T}(M)$, where X is a vector field and F is a tensor field of type $(1, 1)$ on M. We define, for every $\lambda \in N(p, r)$, the *λ-lift* $D^{(\lambda)}$ of D to $T_p^r M$ by

$$D^{(\lambda)} = \mathcal{L}_X(\lambda) + i_F(\lambda)$$

Taking into account Propositions 1.2 and 1.4, we have

PROPOSITION 2.2 $\quad D^{(\lambda)} f^{(\mu)} = (Df)^{(\mu - \lambda)}$

$$D^{(\lambda)}Y^{(\mu)}=(DY)^{(\lambda+\mu)},$$

for any function f and any vector field Y on M, and $\mu\in N(p,r)$.

The *complete lift* D^C of D to T_p^rM is defined by

$$D^C=\mathcal{L}_{X^C}+i_{F^C}$$

Particularizing the Proposition 2.2 to this case, we obtain

$$(2.1)\qquad D^Cf^{(\mu)}=(Df)^{(\mu)}$$
$$D^CY^C=(DY)^C,$$

for any function f and any vector field Y on M. As a direct consequence of Propositions 2.1 and 2.2 and taking into account the above remark, we easily deduce

THEOREM 2.3. *The mapping $D\longrightarrow D^C$ is a Lie algebra homomorphism of $\mathcal{D}(M)$ into $\mathcal{D}(T_p^rM)$.*

REMARK. The mapping $D\longrightarrow D^{(\lambda)}$, $\lambda\neq(0,\dots,0)$, is not a Lie algebra homomorphism because

$$[X^{(\lambda)},Y^{(\lambda)}]=[X,Y]^{(2\lambda)},$$

taking into account (1.1).

Next, we shall consider the lifts of covariant differentiations. Let ∇ be a linear connection on M. Then, the covariant differentiation ∇_X with respect to a vector field X on M is a derivation of $\mathcal{T}(M)$. Since $\nabla_Xf=Xf$ for any function f on M, we have the decomposition

$$\nabla_X=\mathcal{L}_X+i_F$$

where F is a tensor field of type (1,1) on M. We notice that $FY=\nabla_XY-[X,Y]=\hat{\nabla}_YX$, that is, $F=\hat{\nabla}X$, where $\hat{\nabla}$ denotes the opposite connection of ∇. Let $(\nabla_X)^{(\lambda)}$ be the λ-lift of ∇_X to T_p^rM. Taking into account the proposition 2.2, we have

$$(2.2)\qquad (\nabla_X)^{(\lambda)}f^{(\mu)}=(\nabla_Xf)^{(\mu-\lambda)}=(Xf)^{(\mu-\lambda)}$$
$$(\nabla_X)^{(\lambda)}Y^{(\mu)}=(\nabla_XY)^{(\lambda+\mu)},$$

for any function f and any vector field Y on M, and $\mu\in N(p,r)$.

On the other hand, we can consider the complete lift ∇^C of ∇ to T_p^rM and the covariant differentiation $\nabla_X{}^C(\lambda)$ with respect to the λ-lift $X^{(\lambda)}$ of X to T_p^rM. Taking into account the propositions 1.2 and 1.5, we have

$$(2.3)\qquad \nabla_X{}^C(\lambda)f^{(\mu)}=X^{(\lambda)}f^{(\mu)}=(Xf)^{(\mu-\lambda)}$$

$$\nabla_X{}^C(\lambda)\,Y^{(\mu)} = (\nabla_X Y)^{(\lambda+\mu)}$$

for any function f and any vector field Y on M, and $\mu \in N(p,r)$.

PROPOSITION 2.4. *$(\nabla_X)^{(\lambda)} = \nabla_X{}^C(\lambda)$ for any vector field X on M and $\lambda \in N(p,r)$. In particular, $(\nabla_X)^C = \nabla_{X^C}{}^C$.*

REMARK. The results contained in this paper holds good, making small changes, for the r-frame bundle of M. On the other hand, they can be extended to the case of bundles of infinitely near points [5].

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