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ORDERS OF $\mathrm{END}_D(M)$ AND $U(\mathrm{END}_D(M))$ FOR $f.g.$ TORSION MODULE M OVER $p.i.d.$ D

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1. Introduction

In this paper we find the orders of the endomorphism ring $\mathrm{End}_D(M)$ and its unit group $U(\mathrm{End}_D(M))$ for the finitely generated torsion module over the principal ideal domain D whose residue class fields modulo prime ideals in D are all finite.

If $M(\neq 0)$ is a finitely generated module over the $p.i.d.$ D, M is the direct sum of cyclic modules: $M=Dz_1\oplus\cdots\oplus Dz_s$ such that

$$\mathrm{ann}\ z_1\supseteq \mathrm{ann}\ z_2\supseteq\cdots\supseteq \mathrm{ann}\ z_s,\quad \mathrm{ann}\ z_i\neq D$$

and the ideals ann z_i are unique for module M. If the module M is the torsion module and if we put ann $z_i=(d_i)$, d_i are nonzeros, nonunits and $d_1|d_2|\cdots|d_s$. We call $d_1, d_2, \cdots, d_s$ the *invariant factors* of the torsion module M. If $d_i=p_{i,1}{}^{e_{i,1}}p_{i,2}{}^{e_{i,2}}\cdots p_{i,t_i}{}^{e_{i,t_i}}$ is the prime-power decomposition of d_i, then there exist $x_{i,1}, x_{i,2}, \cdots, x_{i,t_i}\in M$ such that

$$Dz_i=Dx_{i,1}\oplus\cdots\oplus Dx_{i,t_i},\quad \mathrm{ann}x_{i,j}=(p_{i,j}{}^{e_{i,j}}).$$

We call $p_{i,j}{}^{e_{i,j}}(1\leq i\leq s,\ 1\leq j\leq t_i)$ the *elementary divisors* of M.

Now let D be a principal ideal domain. We denote the cardinal number of the residue class ring $D/(a)$ modulo ideal $(a)\subseteq D$ by $N(a)$ (the *norm* of a). Then it is easily verified that $N(a)N(b)=N(ab)$ for any $a, b\in D$, and $N(ab)$ is finite if and only if $N(a)$ and $N(b)$ are finite.

The following lemma can be found in [2].

LEMMA 1.1. *Let p be a prime element in $p.i.d.$ D and let e be a positive integer. Suppose that $N(p)$ is finite. Then the order of the general linear group* $\mathrm{GL}(n, D/(p^e))$ *is given by*

$$N(p)^{en^2}\Big(1-\frac{1}{N(p)}\Big)\Big(1-\frac{1}{N(p)^2}\Big)\cdots\Big(1-\frac{1}{N(p)^n}\Big).$$

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2. Endomorphism ring $\mathrm{End}_D(M)$

We consider the problem of explicitly determining the ring $\mathrm{End}_D(M)$ of endomorphisms of finitely generated module M over $p.i.d. D.$ and we will find the order of $\mathrm{End}_D(M)$ when M is the torsion module.

THEOREM 2.1. *Let $M=Dz_1\oplus\cdots\oplus Dz_s$ where the order ideals* ann $z_i=(d_i)$ *satisfy* ann $z_1\supseteq$ *and* $z_2\supseteq\cdots\supseteq$ ann z_s *and* ann $z_i\neq 0$ *for $i\leq r$ but* ann $z_i=0$ *if $i>r$. Then the ring $\mathrm{End}_D(M)$ is isomorphic to R/K where R is the ring of matrices $A\in\mathrm{Mat}_s(D)$ of the form*

$$(*)\quad A=\begin{pmatrix} a_{11} & a_{12}\cdots\cdots\cdots a_{1r} & a_{1r+1}\cdots\cdots a_{1s} \\ a_{21}d_2/d_1 & a_{22}\cdots\cdots\cdots a_{2r} & a_{2r+1}\cdots\cdots a_{2s} \\ \cdots & \cdots & \cdots \\ a_{r1}d_r/d_1 & a_{r2}d_r/d_2\cdots a_{rr} & a_{rr+1}\cdots\cdots a_{rs} \\ 0 & 0\cdots\cdots\cdots 0 & a_{r+1r+1}\cdots a_{r+1s} \\ \cdots & \cdots & \cdots \\ 0 & 0\cdots\cdots\cdots 0 & a_{sr+1}\cdots\cdots a_{ss} \end{pmatrix},\quad a_{ij}\in D$$

whose lower left-hand corner consists of 0's, all the indicated a_{ij} are arbitrary, and the (i,j) entry for $j<i\leq r$ is $a_{ij}d_i/d_j$, and K is the ideal in R of the matrices of the form

$$(**)\quad B=\begin{pmatrix} b_{11}d_1 & b_{12}d_1\cdots\cdots b_{1r}d_1\cdots\cdots b_{1s}d_1 \\ b_{21}d_1 & b_{22}d_2\cdots\cdots b_{2r}d_2\cdots\cdots b_{2s}d_2 \\ \cdots & \cdots \\ b_{r1}d_1 & b_{r2}d_2\cdots\cdots b_{rr}d_r\cdots\cdots b_{rs}d_r \\ 0 & 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots & \cdots \\ 0 & 0\cdots\cdots 0\cdots\cdots 0 \end{pmatrix},\quad b_{ij}\in D$$

whose (i,j) entry is 0 if $i>r$, $b_{ij}d_i$ if $i\leq r$ and $i\leq j\leq s$, $b_{ij}d_j$ if $j<i\leq r$, and all the indicated b_{ij} are arbitrary.

Proof. Let $\eta\in\mathrm{End}_D(M)$ and suppose $\eta(z_j)=w_j\in M$, $1\leq j\leq s$.

Then if $x\in M$, $x=\sum_{i=1}^{s} a_jz_j$, $a_j\in D$ and hence

$$\eta(x)=\eta(\sum a_jz_j)=\sum a_j\eta(z_j)=\sum a_jw_j.$$

This shows that η is determined by its effect on the generators z_i of M. Moreover, $d_jw_j=d_j\eta(z_j)=\eta(d_jz_j)=0$, which shows that ann $w_j\supseteq$ann z_j, so if ann $w_j=(g_j)$, then g_j is arbitrary if $j>r$, and $g_j|d_j$ if $j\leq r$.

Conversely, suppose that for all j we pick an element $w_j\in M$ such that $\mathrm{ann}\ w_j \supseteq \mathrm{ann}\ z_j$. Suppose $x \in M$ and $x = \sum a_j z_j = \sum b_j z_j$ are two representations of x. Then we have $a_j - b_j \in \mathrm{ann}\ z_j$. So $a_j - b_j \in \mathrm{ann}\ w_j$ and consequently $\sum a_j w_j = \sum b_j w_j$. This shows that $\eta : \sum a_j z_j \longrightarrow \sum a_j w_j$ is a map of M into M. Direct verification shows that $\eta \in \mathrm{End}_D(M)$.

Our result is the following. We have a bijection $\eta \longrightarrow (w_1, \cdots, w_s)$ of the ring $\mathrm{End}_D(M)$ onto the set of s-tuples of elements of M satisfying $\mathrm{ann}\ w_j \supseteq \mathrm{ann}\ z_j$. We now write $w_j = \sum_{i=1}^{s} c_{ij} z_i$, $c_{ij} \in D$ and we associate with the s-tuple $(w_1, \cdots, w_s)$ the matrix $A = [c_{ij}]$ in the ring $\mathrm{Mat}_s(D)$ of $s \times s$ matrices with entries in D. This matrix may not be uniquely determined since c_{ij} may be replaced by c_{ij}' such that $c_{ij}' \equiv c_{ij} (\mathrm{mod}\ d_i)$ if $i \leq r$. This is the only alternation which can be made without changing the w_j. The condition that $\mathrm{ann}\ w_j \supseteq \mathrm{ann}\ z_j$ is equivalent to

$$c_{ij} d_j \equiv 0 (\mathrm{mod}\ d_i).$$

This, of course, means that there exists $e_{ij} \in D$ such that $c_{ij} d_j = d_i e_{ij}$. Hence the above condition is equivalent to the following condition on the matrix A: there exists a matrix $E = [e_{ij}] \in \mathrm{Mat}_s(D)$ such that

$$A\ \mathrm{diag}\{d_1, d_2, \cdots, d_s\} = \mathrm{diag}\{d_1, d_2, \cdots, d_s\} E.$$

The set R of matrices A satisfying the above condition is a subring of $\mathrm{Mat}_s(D)$. Any $A = [c_{ij}] \in R$ determines an $\eta \in \mathrm{End}_D(M)$ such that $\eta(z_j) = \sum c_{ij} z_i$. It is easy to verify that the map $A \longrightarrow \eta$ is an epimorphism of R onto $\mathrm{End}_D(M)$. It is clear that $\eta = 0$ if and only if $c_{ij} \equiv 0 (\mathrm{mod}\ d_i)$ for $A = [c_{ij}]$. Hence the kernel K of our homomorphism is the set of matrices A such that

$$A = \mathrm{diag}\{d_1, d_2, \cdots, d_s\} Q$$

where $Q \in \mathrm{Mat}_s(D)$, and $\mathrm{End}_D(M) \cong R/K$.

Now a more explicit determination of the ring of matrices R can be made if we make use of the conditions on d_i that $d_i | d_j$ if $i \leq j \leq r$, and $d_i = 0$ if $i > r$. The conditions $c_{ij} d_j \equiv 0 (\mathrm{mod}\ d_i)$ then imply:

c_{ij} is arbitrary if $i \leq j$ since in this case $d_j \equiv 0 (\mathrm{mod}\ d_i)$;

$c_{ij} = 0$ if $i \geq r$ and $j \leq r$ since in this case $d_i = 0$ and $d_j \neq 0$;

c_{ij} is arbitrary if $i, j > r$ since $d_i = d_j = 0$ in this case;

$c_{ij} \equiv 0 (\mathrm{mod}\ d_i / d_j)$ if $j < i \leq r$.

Therefore changing the notation slightly we see that the matrix A of R has the above form $(*)$ in the theorem.

Now let $A = [c_{ij}] \in K \subseteq R$ and A is the matrix of the form$(*)$. Then every entry of i-th row of A is a multiple of d_i, so if $i > r$ every (i, j)

entry is 0 since $d_i=0$, and when $j<i\le r$, $d_i|(a_{ij}d_i/d_j)$ if and only if $d_j|a_{ij}$.

Hence the matrix of K is the form(**) in the theorem.

THEOREM 2.2. *Let M be the finitely generated torsion module over the p.i.d. D and let $d_1, d_2, \cdots, d_r$ be the invariant factors of M such that $d_1|d_2|\cdots|d_r$. If $N(d_r)$ is finite, then the order of* $\mathrm{End}_D(M)$ *is given by*

$$\prod_{j=1}^{r} N(d_j)^{2r-2j+1}.$$

Proof. Since M is the tosion module $r=s$ in Theorem 2.1, and every $\eta\in\mathrm{End}_D(M)$ is represented by a matrix $A\in R$ of them form

$$A=\begin{pmatrix} a_{11} & a_{12}\cdots\cdots\cdots a_{1r} \\ a_{21}d_2/d_1 & a_{22}\cdots\cdots\cdots a_{2r} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{r1}d_r/d_1 & a_{r2}d_r/d_2 \cdots a_{rr} \end{pmatrix}$$

Let T_i be a complete set of residues modulo d_i for each $i(1\le i\le r)$. Then by Theorem 2.1 any a_{ij} can be replaced by the unique a_{ij}' in T_i, *or* T_j. Hence we may assume $a_{ij}\in T_i$ if $i\le j$. Similarly, we may assume $a_{ij}\in T_j$ if $i>j$. Matrices $A\in R$ satisfying these conditions will be called *normalized* by the sets $T_1, T_2, \cdots, T_r$. It is clear that the map $A\longrightarrow\eta$ restricted to normalized matrices of R is a bijection onto $\mathrm{End}_D(M)$. Therefore the order of $\mathrm{End}_D(M)$ is the same as the number of the normalized matrices. Since $N(d_r)=|T_r|$ is finite and $d_1|d_2|\cdots|d_r$, all $N(d_j)=|T_j|$ are finite. Hence

$$\begin{aligned}|\mathrm{End}_D(M)| &= \prod_{i\le j} N(d_i)\cdot\prod_{i>j} N(d_j)\\ &= N(d_1)^r N(d_2)^{r-1}\cdots N(d_{r-1})^2 N(d_r)\times\\ &\quad N(d_1)^{r-1}N(d_2)^{r-2}\cdots N(d_{r-1})\\ &= N(d_1)^{2r-1}N(d_2)^{2r-3}\cdots N(d_{r-1})^3 N(d_r)\\ &= \prod_{j=1}^{r} N(d_j)^{2r-2j+1}.\end{aligned}$$

3. Unit group $U(\mathrm{End}_D(M))$

Let M be a finitely generated torsion module over a *p.i.d.* D, and for each prime element $p\in D$, let $M(p)=\{z\in M|\operatorname{ann} z=(p^n)$ for some $n\ge1\}$. Then there exist finite number of nonassociate prime elements $p_1, \cdots, p_r$ in D such that

$$M=M(p_1)\oplus\cdots\oplus M(p_r).$$

In this case, it is easy to verify that

$$\mathrm{End}_D(M)\cong \mathrm{End}_D(M(p_1))\oplus\cdots\oplus \mathrm{End}_D(M(p_r)),$$
$$U(\mathrm{End}_D(M))\cong U(\mathrm{End}_D(M(p_1)))\times\cdots\times U(\mathrm{End}_D(M(p_r)))$$

Now assume that every element of the module M has order ideal which is a power of the fixed prime element p in D. Then M is a direct sum of cyclic D-modules of order ideals $(p^{n_1}),\cdots,(p^{n_r})$ respectively, where $1\leq n_1\leq n_2\leq\cdots\leq n_r$, that is,

$$M=Dz_1\oplus\cdots\oplus Dz_r,$$
$$\mathrm{ann}\ z_i=(p^{n_i}),\ 1\leq i\leq r.$$

Then by Theorem 2.1 $\mathrm{End}_D(M)$ is isomorphic onto the ring R/K where R is the ring of matrices of the form

$$(*)\qquad A=\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots\cdots a_{1r} \\ p^{n_2-n_1}a_{21} & a_{22} & \cdots\cdots\cdots a_{2r} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ p^{n_r-n_1}a_{r1} & p^{n_r-n_2}a_{r2} & \cdots a_{rr} \end{pmatrix},\quad a_{ij}\in D$$

and K is the ideal of R of matrices of the form

$$(**)\qquad B=\begin{pmatrix} p^{n_1}b_{11} & p^{n_1}b_{12}\cdots\cdots p^{n_1}b_{1_r} \\ p^{n_1}b_{21} & p^{n_2}b_{22}\cdots\cdots p^{n_2}b_{2_r} \\ \cdots\cdots & \cdots\cdots \\ p^{n_1}b_{r_1} & p^{n_2}b_{r_2}\cdots\cdots p^{n_r}b_{rr} \end{pmatrix},\quad b_{ij}\in D.$$

LEMMA 3.1. *Let p be a prime element of $p.i.d.D$, $n_1,n_2,\cdots,n_r$ be positive integers such that $n_1\leq n_2\leq\cdots\leq n_r$. Let R be the ring of matrices of the above form $(*)$ and let K be the ideal of R of matrices of the above form $(**)$. Then for an element $\bar{A}=A+K\in R/K$, $\bar{A}\in U(R/K)$ if and only if* $(\mathrm{Det}(A),p)=1$.

Proof. If $\bar{A}\in U(R/K)$ there exists $\bar{B}\in U(R/K)$ such that $\overline{AB}=\bar{A}\bar{B}=\bar{B}\bar{A}=\bar{I}$, i.e.,

$$AB=I+\begin{pmatrix} p^{n_1}b_{11} & p^{n_1}b_{12}\cdots\cdots p^{n_1}b_{1r} \\ p^{n_1}b_{21} & p^{n_2}b_{22}\cdots\cdots p^{n_2}b_{2r} \\ \cdots\cdots & \cdots\cdots \\ p^{n_1}b_{r1} & p^{n_2}b_{r2}\cdots\cdots p^{n_r}b_{rr} \end{pmatrix}$$

Therefore $(\mathrm{Det}(A))(\mathrm{Det}(B))=\mathrm{Det}(AB)=1+pc$ for some $c\in D$ hence $(\mathrm{Det}(A),p)=1$. Conversely suppose $(\mathrm{Det}(A),p)=1$ for $\bar{A}\in R/K$. We can easily verify that the adjoint matrix $\mathrm{Adj}(A)$ of A is also an element of R. Then

$$\mathrm{Adj}(A)A = A\mathrm{Adj}(A) = \mathrm{Det}(A)I$$

and there exist $u, v \in D$ such that $\mathrm{Det}(A)u - p^{n_r}v = 1$, since $(\mathrm{Det}(A), p^{n_r}) = 1$. Then,

$$\begin{aligned} u\mathrm{Adj}(A)A = A(u\ \mathrm{Adj}(A)) &= u\mathrm{Det}(A)I \\ &= (1+p^{n_r}v)I = I + p^{n_r}vI,\ p^{n_r}vI \in K. \end{aligned}$$

Therefore $\overline{A}\ \overline{u\mathrm{Adj}(A)} = \overline{u\mathrm{Adj}(A)}\overline{A} = \overline{I}$, so $\overline{A} \in U(R/K)$.

THEOREM 3.2. *Let M be a finitely generated torsion module over p.i.d. D which has the elementary divisors $p^{n_1}, p^{n_2}, \cdots, p^{n_r}$, $1 \leq n_1 \leq \cdots \leq n_r$, for some fixed prime element p in D. Assume that*

$$1 \leq n_1 = \cdots = n_{k(1)} < n_{k(1)+1} = \cdots = n_{k(1)+k(2)} < \cdots$$
$$\cdots < n_{k(1)+\cdots+k(s-1)+1} = \cdots = n_{k(1)+\cdots+k(s)} = n_r.$$

Then if $N(p)$ is finite, the order of $U(End_D(M))$ is given by

$$N(p)^{\alpha} \prod_{i=1}^{s} Q_{k(i)}(p)$$

where

$$\alpha = \sum_{i=1}^{s} \sum_{j=1}^{s} n_{k(1)+\cdots+k(\min(i,j))} k(i)k(j),$$

$$Q_{k(i)}(p) = \left(1 - \frac{1}{N(p)}\right)\left(1 - \frac{1}{N(p)^2}\right)\cdots\left(1 - \frac{1}{N(p)^{k(i)}}\right)$$

Proof. We denote $n_{k(1)} = l_1, n_{k(1)+k(2)} = l_2, \cdots, n_{k(1)+\cdots+k(s)} = l_s$. Then by the above remark any $\eta \in \mathrm{End}_D(M)$ is represented by the matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \cdots\cdots\cdots A_{1s} \\ p^{l_2-l_1}A_{21} & A_{22} & A_{23} \cdots\cdots\cdots A_{2s} \\ p^{l_3-l_1}A_{31} & p^{l_3-l_2}A_{32} & A_{33} \cdots\cdots\cdots A_{3s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ p^{l_s-l_1}A_{s1} & p^{l_s-l_2}A_{s2} & p^{l_s-l_3}A_{s3} \cdots A_{ss} \end{pmatrix}$$

where A_{ij} is a $k(i) \times k(j)$ matrix whose entries are in a prescribed complete set of residues modulo p^{l_i} if $i \leq j$ and modulo p^{l_i} if $i > j$, respectively. Thus A_{ij} is regarded as a matrix in $\mathrm{Mat}_{k(i)\times k(j)}(D/(p^{l_i}))$ if $i \leq j$ and it is regarded as a matrix in $\mathrm{Mat}_{k(i)\times k(j)}(D/p^{l_j}))$ if $i > j$.

Now $U(\mathrm{End}_D(M)) \cong U(R/K)$, so $|U(\mathrm{End}_D(M))| = |U(R/K)|$.

We may write $A=A_0+P$ where

$$A_0=\begin{pmatrix} A_{11} & A_{12}\cdots\cdots A_{12} \\ 0 & A_{22}\cdots\cdots A_{2s} \\ \cdots\cdots & \cdots\cdots \\ 0 & 0 \;\cdots\cdots A_{ss} \end{pmatrix}, \quad P=\begin{pmatrix} 0 & 0\cdots\cdots\cdots 0 \\ p^{l_2-l_1}A_{21} & 0\cdots\cdots\cdots 0 \\ \cdots\cdots & \cdots\cdots \\ p^{l_s-l_1}A_{s1} & p^{l_s-l_2}A_{s2}\cdots 0 \end{pmatrix}$$

Then utilizing Lemma 3.1 for $\bar{A}\in R/K$, we have

$$\bar{A}\in U(R/K) \Longleftrightarrow (\mathrm{Det}(A), p)=1 \Longleftrightarrow (\mathrm{Det}(A_0), p)=1 \Longleftrightarrow (\mathrm{Det}(A_{ii}), p)=1 \text{ for all } i,$$

since p is a prime element in D, $1\leq l_1<l_2<\cdots<l_s$ and $\mathrm{Det}(A_0)=\prod_{i=1}^{s}\mathrm{Det}(A_{ii})$. Thus if $\bar{A}\in U(R/K)$, every $k(i)\times k(i)$ matrix A_{ii} is regarded as a matrix in the general linear gorup $\mathrm{GL}(k(i), D/p^{l_i}))$.

Now suppose that $N(p)$ is finite. Then, since the order of group $\mathrm{GL}(k(i), D/(p^{l_i}))$ is given by $N(p)^{l_ik(i)^2}\,Q_{k(i)}(p)$ where $Q_{k(i)}(p)=\left(1-\frac{1}{N(p)}\right)\left(1-\frac{1}{N(p)^2}\right)\cdots\left(1-\frac{1}{N(p)^{k(i)}}\right)$ by Lemma 1.1, and the number of choices for A_{ij} is $N(p)^{l_ik(i)k(j)}$ when $i<j$ and $N(p)^{l_jk(i)k(j)}$ when $i>j$ respectively, it follows that the order of $U(R/K)$ is given by

$$\prod_{i<j}N(p)^{l_ik(i)k(j)}\prod_{i>j}N(p)^{l_jk(i)k(j)}\prod_{i=1}^{s}N(p)^{l_ik(i)^2}\,Q_{k(i)}(p)=N(p)^{\alpha}\prod_{i=1}^{s}Q_{k(i)}(p)$$

where

$$\alpha=\sum_{i<j}l_ik(i)k(j)+\sum_{i>j}l_jk(i)k(j)+\sum_{i=1}^{s}l_ik(i)^2$$
$$=\sum_{i=1}^{s}\sum_{j=1}^{s}l_{\min(i,j)}k(i)k(j)=\sum_{i=1}^{s}\sum_{j=1}^{s}n_{k(1)+\cdots+k(\min(i,j))}k(i)k(j).$$

THEOREM 3.3. *Let M be a finitely generated torsion module over a p. i. d. D which has elementary divisors*

$$p_\lambda^{n_{\lambda,1}}, p_\lambda^{n_{\lambda,2}}, \cdots, p_\lambda^{n_{\lambda,r_\lambda}}, \quad 1\leq\lambda\leq t,$$

where $p_1, p_2, \cdots, p_t$ are nonassociate prime elements in D such that $N(p_1)$, $N(p_2), \cdots, N(p_t)$ are all finite and $1\leq n_{\lambda,1}\leq\cdots\leq n_{\lambda,r_\lambda}$ for all λ.

Assume that

$$1\leq n_{\lambda,1}=\cdots=n_{\lambda,k(\lambda,1)}<n_{\lambda,k(\lambda,1)+1}=\cdots=n_{\lambda,k(\lambda,1)+k(\lambda,2)}<\cdots$$
$$\cdots<n_{\lambda,k(\lambda,1)+\cdots+k(\lambda,s_\lambda-1)+1}=\cdots=n_{\lambda,k(\lambda,1)+\cdots+k(\lambda,s_\lambda)}=n_{\lambda,r_\lambda}.$$

Then the order of $U(\mathrm{End}_D(M))$ is given by

$$\prod_{\lambda=1}^{t} N(p_\lambda)^{\alpha_\lambda}\left(\prod_{j=1}^{s_\lambda} Q_{\lambda,k(\lambda,j)}(p_\lambda)\right)$$

where

$$\alpha_\lambda=\sum_{i=1}^{s_\lambda}\sum_{j=1}^{s_\lambda} n_{\lambda,k(\lambda,1)+\cdots k(\lambda,\min(i,j))}k(\lambda,i)k(\lambda,j),$$

$$Q_{\lambda,k(\lambda,j)}(p_\lambda)=\left(1-\frac{1}{N(p_\lambda)}\right)\left(1-\frac{1}{N(p_\lambda)^2}\right)\cdots\left(1-\frac{1}{N(p_\lambda)^{k(\lambda,j)}}\right).$$

Proof. Let $M(p_\lambda)=\{z\in M \mid \text{ann } z=(p_\lambda^m) \text{ for some integer } m\geq 1\}$. Then

$$M=M(p_1)\oplus M(p_2)\oplus\cdots\oplus M(p_t),$$

$$\text{End}_D(M)\cong\text{End}_D(M(p_1))\oplus\cdots\oplus\text{End}_D(M(P_t)),$$

$$U(\text{End}_D(M))\cong U(\text{End}_D(M(p_1)))\times\cdots\cdots\times U(\text{End}_D(M(p_t))),$$

and $p_\lambda^{n_{\lambda,1}},\cdots,p_\lambda^{n_{\lambda,r_\lambda}}$ are elementary divisors of $M(p_\lambda)$ for each λ. Therefore by Theorem 3.2, we have the desired order of the unit group $U(\text{End}_D(M))$.

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