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A REPRESENTATION OF $E(X, x_0, G)$ IN TERMS OF $G(X, x_0)$

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F. Rhodes [2] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group of a topological space X and showed that $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0)\times G$ if (G, G) admits a family of preferred paths at e. D.H. Gottlieb [1] introduced the evaluation subgroup $G(X, x_0)$ of the fundamental group of a topological space X. The author [4] introduced the evaluation subgroup $E(X, x_0, G)$ of the fundamental group of a transformation group as a generalization of the evaluation subgroup $G(X, x_0)$.

In this paper, we give necessary and sufficient conditions for $E(X, x_0, G)$ $(\sigma(X, x_0, G))$ to be isomorphic to $G(X, x_0)\times G$ $(\pi_1(X, x_0)\times G)$.

Let (X, G, π) be a transformation group, where X is a path connected space with x_0 as base point. Given any element g of G, *a path f of order g* with base point x_0 is a continuous map $f: I\longrightarrow X$ such that $f(0)=x_0$ and $f(1)=gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1+g_1f_2$ of order g_1g_2 defined by the equations

$$(f_1+g_1f_2)(s)=\begin{cases} f_1(2s), & 0\leqslant s\leqslant \frac{1}{2} \\ g_1f_2(2s-1), & \frac{1}{2}\leqslant s\leqslant 1. \end{cases}$$

Two paths f and f' of the same order g are said to be ***homotopic*** if there is a continuous map $F: I^2\longrightarrow X$ such that

$$\begin{aligned} F(s,0)&=f(s) && 0\leqslant s\leqslant 1,\\ F(s,1)&=f'(s) && 0\leqslant s\leqslant 1,\\ F(0,t)&=x_0 && 0\leqslant t\leqslant 1,\\ F(1,t)&=gx_0 && 0\leqslant t\leqslant 1.\end{aligned}$$

The homotopy class of a path f of order g was denoted by $[f; g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct,

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even if $g_1x_0=g_2x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by $[f_1 ; g_1]*[f_2 ; g_2]=[f_1+g_1f_2 ; g_1g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the *fundamental group* of (X, G) with base point x_0.

In [1], a homotopy $H: X\times I\longrightarrow X$ is called a *cyclic homotopy* if $H(x, 0)=H(x, 1)=x$. This concept of a topological space is generalized to that of a transformation group. A continuous map $H: X\times I\longrightarrow X$ is called *a homotopy of order g* if $H(x, 0)=x$, $H(x, 1)=gx$, where g is an element of G. If H is a homotopy of order g, then the path $f: I\longrightarrow X$ such that $f(t)=H(x_0, t)$ will be called *the trace of H.* The trace is a path of order g. In particular, if the acting group G is trivial, then a homotopy of order g is a cyclic homotopy.

In [4], the subgroup $E(X, x_0, G)$ was defined by the set of all elements $[f ; g]\in\sigma(X, x_0, G)$ such that f is the trace of a homotopy of order g, where $g\in G$. The evaluation subgroup $G(X, x_0)$ can be identified by $E(X, x_0, \{e\})$.

In [2], a transformation group (X, G) is said to admit *a family K of preferred paths* at x_0 if it is possible to associate with every element g of G a path k_g from gx_0 to x_0 such that the path k_e associated with the identity element e of G is x_0' which is the constant map such that $x_0'(t)=x_0$ for each $t\in I$ and for every pair of elements g, h, the path k_{gh} from ghx_0 to x_0 is homotopic to gk_h+k_g.

DEFINITION 1. A family K of preferred paths at x_0 is called *a family of preferred traces at* x_0 if for every preferred path k_g in K, $k_g\rho$ is the trace of a homotopy of order g, where $\rho(t)=1-t$.

EXAMPLE 1. Let R be the additive group of real numbers. Then the transformation group (R, R) admits a family of preferred traces at 0, where the action for the transformation group is the map $\pi: R\times R\longrightarrow R$ defined by $\pi(s, t)=s+t$.

THEOREM 1. *Let (X, G, π) be a transformation group. If (G, G) admits a family of preferred paths at e, than (X, G) admits a family of preferred traces at x_0.*

Proof. Let H be a family of preferred paths at e in (G, G). Define $K=\{k_g : k_g(t)=h_g(t)(x_0),\ h_g\in H\}$. Then it is easy to show that K is a family of preferred paths at x_0.

Define $F : X\times I \longrightarrow X$ by

$$F(x, t) = \pi(x, h_g\rho(t)).$$

Then F is a homotopy of order g with trace $k_g\rho$. Thus K is a family of preferred traces at x_0.

The converse of Theorem 1 does not hold;

EXAMPLE 2. Let R be the real space, Z be the additive integer group and $\pi : R\times Z \longrightarrow R$ be a map defined by $\pi(r, n) = r+n$. Then (R, Z, π) is a transformation group and it admits a family of preferred traces at 0. Let

$$K = \{k_n \mid k_n \text{ is a path from } n \text{ to } 0 \text{ in } R\}.$$

It is easy to show that K is a family of preferred paths at 0. For each $n\in Z$, define $H : R\times I \longrightarrow R$ by

$$H(r, t) = r + k_n\rho(\mathrm{t}).$$

Then H is a homotopy of order n with trace $k_n\rho$. Thus K is a family of preferred traces at 0. Since Z is discrete, there is no path from n to 0, where n is any nonzero integer. Thus (Z, Z) cannot admit a family of preferred paths at 0.

By Theorem 1 and Example 1, every transformation group (X, R) admits a family of preferred traces at x_0.

LEMMA 2. *Let (X, G) be a transformation group. If k is a trace of a homotopy of order g, then every loop f at x_0 is homotopic to $k+gf+k\rho$.*

Proof. Let $H : X\times I \longrightarrow X$ be a homotopy of order g with trace k and f be a loop at x_0.

Define $F : I\times I \longrightarrow X$ by

$$F(s, t) = \begin{cases} k(4s) & 0\leqslant s\leqslant t/4 \\ H\left(f\left(\dfrac{4s-t}{4-2t}\right), t\right) & \dfrac{t}{4}\leqslant s\leqslant \dfrac{4-t}{4} \\ k\rho(4s-3) & (4-t)/4\leqslant s\leqslant 1. \end{cases}$$

Then F is well defined, $F(s, 0) = f(s)$ and $F(s, 1) = (k+gf+k\rho)(s)$.

DEFINITION 2. A family K of preferred paths at x_0 is called *a family of preferred strong paths at x_0* if for each loop f at x_0 and each k_g in

K, f is homotopic to $k_g\rho+gf+k_g$.

REMARK. By Lemma 2, every family of preferred traces is a family of preferred strong paths.

THEOREM 3. *A transformation group* (X, G) *admits a family of preferred traces at* x_0 *if and only if* $E(X, x_0, G)$ *is a split extention of* $G(X, x_0)$ *by* G.

Proof. Suppose (X, G) admits a family $K=\{k_g : g \in G\}$ of preferred traces at x_0. Consider the sequence

$$0 \longrightarrow G(X, x_0) \overset{i_G}{\longrightarrow} E(X, x_0, G) \overset{j_G}{\longrightarrow} G \longrightarrow 0$$

where $i_G[f]=[f : e]$ and $j_G[f : g]=g$. Since i_G is a monomorphism and j_G is an epimorphism, the sequence is a short exact sequence. Define $\phi : G \longrightarrow E(X, x_0, G)$ by $\phi(g)=[k_g\rho : g]$. Then ϕ is a homomorphism, for

$$\begin{aligned}\phi(g_1g_2)&=[k_{g_1g_2}\rho \,;\, g_1g_2]=[(g_1k_{g_2}+k_{g_1})\rho \,;\, g_1g_2]\\&=[k_{g_1}\rho+g_1k_{g_2}\rho \,;\, g_1g_2]=[k_{g_1}\rho \,;\, g_1]*[k_{g_2}\rho \,;\, g_2]\\&=\phi(g_1)*\phi(g_2).\end{aligned}$$

By definition, we have $j_G\phi=1_G$. Thus $E(X, x_0, G)$ is a split extension of $G(X, x_0)$ by G.

Conversely, suppose $E(X, x_0, G)$ is a split extension of $G(X, x_0)$ by G. Then there is a monomorphism $\Psi : G \longrightarrow E(X, x_0, G)$ such that $j_G\Psi=1_G$. Let $H=\{f_g \mid f_g\rho$ is a representative path of $\Psi(g)\}$. Then H is a family of preferred traces at x_0. Since $\Psi(e)=[x_0' : e]$, $\Psi(g_1g_2)=\Psi(g_1)*\Psi(g_2)$, $f_{g_1g_2}$ is homotophic to $(g_1f_{g_2}+f_{g_1})$ and $f_e=x_0'$.

In [4] the author showed that if a transformation group (X, G) admits a family of preferred traces at x_0, then $E(X, x_0, G)$ is isomorphic to $G(X, x_0)\times G$, but the proof was not complete.

THEOREM 4. *A transformation group* (X, G) *admits a family of preferred traces at* x_0 *if and only if there is an isomorphism* $\phi : E(X, x_0, G) \longrightarrow G(X, x_0)\times G$ *such that the diagram commutes*

$$\begin{array}{ccccc} & & E(X, x_0, G) & & \\ 0\longrightarrow G(X, x_0) & \nearrow\!\!\searrow & \downarrow \phi & \searrow\!\!\nearrow & G\longrightarrow 0 \\ & & G(X, x_0)\times G & & \end{array}$$

Proof. Let $K=\{k_g : g\in G\}$ be a family of preferred traces at x_0. Define $\phi : E(X, x_0, G) \longrightarrow G(X, x_0)\times G$ by

$$\phi([f ; g])=([f+k_g], g)$$

Since $[f ; g]\in E(X, x_0, G)$, there exists a homotopy $H: X\times I\longrightarrow X$ of order g with trace f. $k_g\rho$ is a trace of a homotopy $J: X\times I\longrightarrow X$ of order g. Define $F : X\times I\longrightarrow X$ by

$$F(x, t)=\begin{cases} H(x, 2t), & 0\leqslant t\leqslant 1/2 \\ J(x, 2(1-t)), & 1/2\leqslant t\leqslant 1. \end{cases}$$

Then F is a cyclic homotopy with trace $f+k_g$. Thus $[f+k_g]$ belongs to $G(X, x_0)$. Let $[f ; g]=[f' ; g]$. Since f is homotopic to f', $f+k_g$ is also homotopic to $f'+k_g$. Thus ϕ is well defined.

Suppose $\phi([f ; g])=\phi([f' ; g])$. Then $f+k_g$ is homotopic to $f'+k_g$. This implies that $f(=f+k_g+k_g\rho)$ is homotopic to $f'(=f'+k_g+k_g\rho)$. Therefore ϕ is injective. For any element $([f], g)\in G(X, x_0)\times G$, there exists an element $[f+k_g\rho ; g]$ in $E(X, x_0, G)$ such that

$$\phi([f+k_g\rho ; g])=([f+k_g\rho+k_g], g)=([f], g).$$

Thus ϕ is a bijection.

Next we must show that ϕ is a homomorphism. Let $[f_1 ; g_1]$ and $[f_2 ; g_2]$ be elements of $E(X, x_0, G)$. Then

$$\begin{aligned}\phi([f_1 ; g_1]*[f_2 ; g_2]) &=\phi([f_1+g_1f_2 ; g_1g_2]) \\ &=([f_1+g_1f_2+k_{g_1g_2}] ; g_1g_2)\end{aligned}$$

while

$$\begin{aligned}\phi([f_1 ; g_1]) \circ \phi[f_2 ; g_2]) &=([f_1+k_{g_1}], g_1) \circ ([f_2+k_{g_2}], g_2) \\ &=([f_1+k_{g_1}+f_2+k_{g_2}], \ g_1g_2).\end{aligned}$$

Since $f_2+k_{g_2}$ is a loop at x_0 and $k_{g_1}\rho$ is a trace of a homotopy of order g_1, $f_2+k_{g_2}$ is homotopic to $k_{g_1}\rho+g_1(f_2+k_{g_2})+k_{g_1}$ by Lemma 2. Therefore we obtain

$$\begin{aligned}f_1+k_{g_1}+f_2+k_{g_2} &\sim f_1+k_{g_1}+k_{g_1}\rho+g_1(f+k_{g_2})+k_{g_1} \\ &\sim f_1+g_1(f_2+k_{g_2})+k_{g_1} \\ &\sim f_1+g_1f_2+k_{g_1g_2}.\end{aligned}$$

Conversely, given a commutative diagram with exact rows and ϕ an isomorphism:

$$
\begin{array}{ccccc}
 & & E(X, x_0, G) & & \\
 & \overset{i_G}{\nearrow} & \downarrow \phi & \overset{j_G}{\searrow} & \\
0 \longrightarrow G(X, x_0) & & & & G \longrightarrow 0 \\
 & \underset{i_1}{\searrow} & & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & \\
 & & G(X, x_0) \times G & &
\end{array}
$$

define $\psi : G \longrightarrow E(X, \pi_0, G)$ to be $\phi^{-1} i_2$. Use the commutativity of the diagram to show $j_G \psi = 1_G$. Then $E(X, x_0, G)$ is a split extension of $G(X, x_0)$ by G. If we apply Theorem 3, (X, G) admits a family of preferred traces at x_0.

COROLLARY 5. *A transformation group (X, G) admits a family of preferred traces at x_0 and G abelian if and only if $0 \longrightarrow G(X, x_0) \longrightarrow E(X, x_0, G) \longrightarrow G \longrightarrow 0$ is a split exact sequence of Z-modules.*

Proof. It is clear by Theorem 4 and $G(X, x_0)$ abelian.

REMARK. Every transformation group (X, R) has the abelian evaluation subgroup $E(X, x_0, R)$.

In [2], F. Rhodes showed that if (G, G) admits a family of preferred paths at e, then $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$. In general, the converse is not true.

THEOREM 6. *A transformation group (X, G) admits a family of preferred strong paths at x_0 if and only if there exists an isomorphism ϕ : $\pi_1(X, x_0) \times G \longrightarrow \sigma(X, x_0, G)$ such that the diagram commutes*

$$
\begin{array}{ccccc}
 & & \sigma(X, x_0, G) & & \\
 & \overset{i_G}{\nearrow} & \uparrow \phi & \overset{j_G}{\searrow} & \\
0 \longrightarrow \pi_1(X, x_0) & & & & G \longrightarrow 0 \\
 & \underset{i_1}{\searrow} & & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & \\
 & & \pi_1(X, x_0) \times G & &
\end{array}
$$

Proof. Only if: It is proved by the same method of Theorem 4.

If: Define $\psi : G \longrightarrow \sigma(X, x_0, G)$ by $\psi = \phi \circ i_2$. If we use the commutatity of the diagram and $\pi_2 \circ i_2 = 1_G$ and ϕ an isomorphism, then we have $j_G \circ \psi = 1_G$. For each $g \in G$, let $\psi(g) = [k_g \rho\ ;\ g]$. Then $K = \{k_g : k_g \rho$ is the representative path of $\psi(g)\}$ is a family of preferred paths at x_0. We have to show that K is a family of preferred strong paths at x_0. Let f is any loop at x_0 and k_g be any element of K. Since

$$([f], e) = ([x_0'], g) \circ ([f], g^{-1})$$

in $\pi_1(X, x_0) \times G$, we have

$$
\begin{aligned}
[f\,;\,e]=i_G([f])=\phi i_1([f])&=\phi([f],e)\\
&=\phi(([x_0'],g)\circ([f],g^{-1}))\\
&=\phi([x_0'],g)*\phi([f],g^{-1})\\
&=[x_0'+k_g\rho;\ g]*[f+k_{g^{-1}}\rho;\ g^{-1}]\\
&=[x_0'+k_g\rho;\ g]*[f+g^{-1}k_g;\ g^{-1}]\\
&=[x_0'+k_g\rho+gf+k_g;\ e]\\
&=[k_g\rho+gf+k_g;\ e]
\end{aligned}
$$

Thus f is homotopic to $k_g\rho+gf+k_g$. Therefore K is a family of preferred strong paths at x_0.

The existence of a family of preferred traces (preferred strong paths) on a transformation group does not depend to base point.

THEOREM 7. *Let (X, G) be a transformation group. If λ is a path from x_0 to x_1, then a family of preferred strong paths at x_0 gives rise to a family of preferred strong paths at x_1 and a family of preferred traces at x_0 induces a family of preferred traces at x_1.*

Proof. Let $K=\{k_g : g\in G\}$ be a family of preferred strong paths at x_0. For each g in G, let $h_g=g\lambda\rho+k_g+\lambda$. It is easy to show that $H=\{h_g : g\in G\}$ is a family of preferred paths at x_1. Let f be any loop at x_1 and h_g be any element of H. Then we have

$$
\begin{aligned}
h_g\rho+gf+h_g&=(g\lambda\rho+k_g+\lambda)\rho+gf+(g\lambda\rho+k_g+\lambda)\\
&=\lambda\rho+k_g\rho+g(\lambda+f+\lambda\rho)+k_g+\lambda.
\end{aligned}
$$

Since $\lambda+f+\lambda\rho$ is a loop at x_0, $\lambda+f+\lambda\rho$ is homotopic to $k_g\rho+g(\lambda+f+\lambda\rho)+k_g$. Thus we obtain that f is homotopic to $h_g\rho+gf+h_g$.

Let $K=\{k_g : g\in G\}$ be a family of preferred traces at x_0 and $h_g=g\lambda\rho+k_g+\lambda$. Since the induced isomorphism λ_* carries $E(X, x_0, G)$ isomorphically on $E(X, x_1, G)$,

$$\lambda_*[k_g\rho\,;\,g]=[\lambda\rho+k_g\rho+g\lambda\,;\,g]=[h_g\rho\,;\,g]$$

belongs to $E(X, x_1, G)$. Thus $H=\{h_g : g\in G\}$ is a family of preferred traces at x_1.

The represetation is natural with respect to change of base point in the sense that the following two diagrams are commutative.

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{\lambda_*} & \sigma(X, x_1, G) \\ \downarrow \phi_K & & \downarrow \phi_H \\ \pi_1(X, x_0)\times G & \longrightarrow & \pi_1(X, x_1)\times G \end{array} \qquad \begin{array}{ccc} E(X, x_0, G) & \xrightarrow{\lambda_*} & E(X, x_1, G) \\ \downarrow \phi_K & & \downarrow \phi_H \\ G(X, x_0)\times G & \longrightarrow & G(X, x_1)\times G \end{array}$$

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