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HIGHER LEVEL SIGNATURES ON VALUATION RINGS

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Introduction

In [1] E. Becker developes a theory of what he calls orderings of higher level over fields. These generalize orderings of a field in such a way that one can generalize many of the usual results in formally real fields. In [8], Kleinstein and Rosenberg show that there is a natural extension of the usual Witt ring of equivalence classes of non-degenerate bilinear forms over a field to the Witt ring of higher level.

In [5], T. Craven defines the Witt ring of higher level over a semilocal ring and extends many of the results by Kleinstein and Rosenberg. In this paper we apply the results by T. Craven to valuation ring A. Thus we can obtain a generalization of the result by Knebusch on the extension of a signature of A to a signature of the quotient field of A. As a Corollary we obtain a result on sum of 2^n-powers problem. We also prove that the Dress' theorem [6] can be generalized to higher level case if A is a local ring with usual conditions. Finally we give two examples convincing us the necessity of the condition A being a valuation ring in our Corollary.

All of the notations and terminologies follow those of T. Craven.

Higher level signature on valuation rings

Let A be a connected semilocal ring with no residue class field having 2 elements. We denote the group of units of A by A^*, write $G_n(A) = A^*/A^{*2^n}$ for the group of units modulo 2^n-powers, and $\langle a\rangle_n$ for aA^{*2^n} where $a\in A^*$.

DEFINITION 1. The Witt ring of level n of A, denoted $W_n(A)$, is the integral groupring $Z[G_n(A)]$ modulo the ideal $J_n(A)$ generated by $\langle 1\rangle_n+\langle -1\rangle_n$ and all elements of the form

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$$(\sum_{0}^{2^n-1}\langle x^i\rangle_n)(\langle 1\rangle_n-\langle \lambda_1{}^{2^n}+\lambda_2{}^{2^n}\cdot x\rangle_n)=(\langle 1\rangle_n-\langle \lambda^{2^n}{}_1+\lambda^{2^n}{}_2\cdot x\rangle_n)$$
$$\prod_{0}^{n-1}(\langle 1\rangle_n+\langle x^{2^i}\rangle_n)$$

whenever x and $\lambda_1+\lambda_2\cdot x$ are both in A^*.

Let C_n denote the group of 2^n–th roots of unity in the complex numbers and U_n denote the ring of integers of the cyclotomic field $Q(C_n)$. If $\sigma : A^* \longrightarrow C_n$ is a homomorphism which extends to $\sigma : W_n(A) \longrightarrow U_n$; that is, when σ is extended to a homomorphism $Z[G_n(A)] \longrightarrow U_n$, its kernel contains $J_n(A)$, we call σ a signature of level n. Let $P=P(\sigma)=\ker(\sigma : A^* \longrightarrow C_n)$, so that $A^*/P \cong C_m$ for some $m \leq n$. In this case we shall say σ is a signature of exact level m. In general, we shall speak of signatures of higher level without specifying n, When A is a field, these definitons coincide with those of [1] and in this case $P(\sigma) \cup \{0\}$ is an ordering of level n.

DEFINITION 2. Let σ be a signature of exact level n. Define $Q(\sigma)=\{\sum \lambda_i^{2^n} a_i \mid \lambda_i \in A,\ \sum \lambda_i A=A,\ a_i \in P(\sigma)\}$.

Note that $Q(\sigma)=P(\sigma)$ if A is a field. Now let $\varphi : A \longrightarrow B$ be a homomorphism between two semilocal rings which satisfy our usual conditions.

DEFINITION 3. Let σ, τ be signatures of exact level n of A and B respectively. Let $\varphi_* : W_n(A) \longrightarrow W_n(B)$ be the homomorphism induced by φ. We say τ is a faithful extension of σ if the homomorphisms

$$W_n(A) \xrightarrow{\varphi_*} W_n(B) \xrightarrow{\tau} U_n \text{ and } W_n(A) \xrightarrow{\sigma} U_n$$

have the same kernel.

In [5], T. Craven defines $S_n(A,B)=\{\sum \lambda_i^{2^n} \varphi(a_i) \mid \lambda_i \in B,\ \sum \lambda_i B=B,\ a_i \in P(\sigma)\}$ for σ and φ as above. Then he shows the signature of exact level n can be extended faithfully to a signature of B if and only if $0 \notin S_n\ (A,B)$.

LEMMA 4. *Let A be a local ring with maximal ideal $\mathfrak{M}$ such that $|A/\mathfrak{M}| \neq 2$. Then every element of $Q(\sigma)$ either lies in $P(\sigma)$ or is a sum of two elements of $P(\sigma)$* [5].

Now let A be a valuation ring with quotient field K, maximal ideal $\mathfrak{M}$. Assume $|A/\mathfrak{M}|\neq 2$ and $2\in A^*$. In [11], M. Knebusch proved any signature of level 1 of A can be extended to a signature of level 1 of K. We have the following complete generalization in our higher level case.

THEOREM 5. *Let A be as above. Then each signature of higher level of A can be extended faithfully to a signature of higher level of K.*

Proof. Let σ be a signature of exact level n of A. By the remark above Lemma 4, it will suffice to show $0\notin S_n(A, K)$. Suppose $\lambda_1^{2^n}a_1+\cdots+\lambda_k^{2^n}a_k=0$ with $\lambda_i\in A$, $a_i\in P(\sigma)$ for $1\leq i\leq k$. If $\lambda_i\in A^*$ for some $1\leq i\leq k$, then $\lambda_1^{2^n}a_1+\cdots+\lambda_k^{2^n}a_k\in Q(\sigma)$. Since A is a local ring, we have $\lambda_1^{2^n}a_1+\cdots+\lambda_k^{2^n}a_k$ is a sum of two elements of $P(\sigma)$ by Lemma 4. Now we have $\lambda_1^{2^n}a_1+\cdots+\lambda_k^{2^n}a_k=a+b=0$ for some $a, b\in P(\sigma)$, then $a=-b$, and hence $1=\sigma(a)=\sigma(-b)=-1$, a contradiction. Therefore we may assume $\lambda_i\notin A^*$ for $1\leq i\leq k$. We denote the (additive) valuation of A by v. If $v(\lambda_1)=\min\{v(\lambda_i)\,|\,1\leq i\leq k\}$, then $v\left(\frac{\lambda_i}{\lambda_1}\right)=v(\lambda_i)-v(\lambda_1)\geq 0$, i. e. $\frac{\lambda_i}{\lambda_1}\in A$ for $1\leq i\leq k$. Our equation reduces to $1\cdot a_1+\lambda'^{2^n}_2a_2+\cdots+\lambda'^{2^n}_k a_k=0$, $\lambda_i'\in A$, $a_i\in P(\sigma)$ for $2\leq i\leq k$. Sine $1\in A^*$, we have a contradiction. Thus $0\notin S_n(A, K)$, and our signature can be extended faithfully to K.

REMARK. If $1+\mathfrak{M}\subset P(\sigma)$, we say σ is compatible with A. If this is the case, we have the following simple proof. Denote the residue field $A/\mathfrak{M}$ by k. Then the character $\bar{\sigma}$ defined by $\bar{\sigma}(\bar{x})=\sigma(x)$ is a well-defined signature of exact level n of k. Since $\bar{\sigma}$ can be lifted faithfully to a signature of K[7], we have proved our theorem.

Now let $\sum(A)$ denote the set of all elements $\sum\lambda_i^{2^n}$ such that $\sum\lambda_iA=A$ together with elements x such that $xy=z$ where y and z are such sums of 2^n-powers. If A is a field then $\sum(A)=\sum A^{2^n}$. For A semilocal and $n=1$, $\sum(A)=\sum A^2$ by the representation criterion for quadratic forms [10].

COROLLARY 6. *Let A and K be as in Theorem 5. Then if a unit element a of A belongs to $\sum K^{2^n}$, a is an element of $\sum(A)$.*

In [3], Kneser, J-L Colliot-Thélène proved that a is already in $\sum A^2$

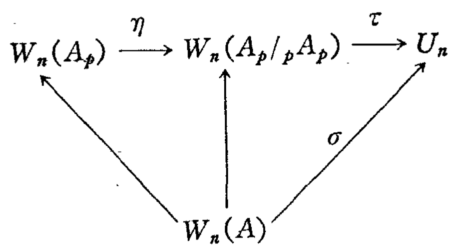
if $a\in A^*\cap\sum F^2$. Since $\sum(A)=\sum A^2$ for $n=1$, Corollary 6 is a generalization of this in higher level case.

Proof. Since $a\in\sum K^{2^n}$, $a\in\cap$ {orderings of level n of K} [2]. Theorem 5 says any sinature of higher level of A can be extended faithfully to K, so that $a\in\cap P(\sigma)$ where σ ranges over all signatures of level n of A. By Theorem 3.7 of [5], $a\in\sum(A)$.

If σ is a usual signature of level 1 of a commutative ring A with $2\in A^*$, Dress' Theorem [6] guarantees the existence of some prime ideal p of A such that σ can be extended to A_p. We can generalize this theorem to higher level signature case for local ring A.

THEOREM 7. *Let A be a local domain with $|A/\mathfrak{M}|\neq 2$ and $2\in A^*$. If σ is a signature of higher level of A, there exists a prime ideal p of A such that σ can be extended faithfully to a signature of higher level of A_p.*

Proof. Since A is a local ring, there exists a prime ideal p of A such that σ can be extended faithfully to the quotient field $A(p)$ of the integral domain A/p [5]. Let τ denote the extension of σ. Since $A(p)=A_p/{}_pA_p$, we have the following diagram.



where η is the natural homomorphism. Since the left triangle commutes, it is clear $\tau\eta$ is an extension of σ on A_p.

Now we give two examples which show the necessity of our condition on A in Corollary 6.

EXAMPLE 8. Let $A_0=R[x,y,z]/(x^2+y^2+z^2)$, $p=(x,y,z)$ and $A=(A_0)_p$. Then A is a local domain of Krull dimension 2. The element $-1\in A^*$ is a sum of two squares in K, but -1 is not a sum of squares in A[3].

EXAMPLE 9. let k be a real-closed field, and S_0 be the set of irreducible polynomials $s \in k[x, y]$ such that s generate a real prime ideal. Let S be the multiplicative set generated by S_0, and $A = S^{-1}(k[x, y])$. Then A is a *PID*. If $f(x, y) = x^3 + (xy - x^2 - 1)^2 f(x, y)$ is a positive semidefinite polynomical with the property that $f^{2r+1} \notin \sum A^2$ for any r. Now we have f is a sum of four squares in $k(x, y)$. If $A' = A[f^{-1}]$, then A' is a *PID*. The unit element $f \notin \sum A'^2$ [4].

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