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SELFCOMMUTATORS IN AN INFINITE SEMIFINITE FACTOR

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1. Introduction.

Let $L(H)$ be the *-algebra of all (bounded linear) operators on a (complex) Hilbert space H of infinite dimension. Let A be a *-subalgebra of $L(H)$. An element $x\in A$ is called a *selfcommutator in* A, if it can be written as the form $x=y^*y-yy^*$ for some $y\in A$.

In the case when H is an infinite dimensional separable Hilbert space, H. Radjavi succeeded to characterize the selfcommutators x in $L(H)$ for the first time, in terms of the spectrum of $\sigma_h(x)$, where $\sigma_h : L(H)\rightarrow L(H)/J_h$ is the quotient mapping and J_h is the largest nontrivial closed (two sided) ideal of $L(H)$ [9]. His characterization of the selfcommutators in $L(H)$ was extended to the case when H is a Hilbert space of arbitrary infinite dimension. Since an infinite type I factor A contained in $L(H)$, A is *-isomorphic onto some $L(K)$, where K is a suitable infinite dimensional Hilbert space, the work [2] is regarded as a characterization of the selfcommutators in an arbitrary I_∞-factor A. But there is still another kind of an infinite semifinite factor, namely, II_∞-factor.

In this article, we are going to present a unified treatment of the characterization of selfcommutators in A comprising both cases when A is either an I_∞-or II_∞-factor in L(H), where H is any given Hilbert space of an arbitrarily infinite dimension.

2. Preliminaries.

LEMMA 1. *Let x_1 and x_2 be two normal operators in a von Neumann algebra A. Suppose that x_1 and x_2 are similar in A, that is, there is an invertible element $a\in A$ such that $a^{-1}x_1a=x_2$. Then, they are unitarily equivalent in A, namely, there is a unitary element $u\in A$ such that u^*x_1u*

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$=x_2$ (*cf. p.* 99 [3] *Problem* 152 *and its Corollary*)

Proof. Let $a=up$ be the left polar decomposition of a with the partial isometry u and $p=(a*a)^{1/2}$. It is well known that $u\in A$. By duplicating the proof of Corollary (p. 99 [3]) in (p. 306[3]), we see that $u^{-1}x_1u=x_2$ with $u\in A$, as desired.

LEMMA 2. *An operator x in von Neumann algebra A is a selfcommutator in A, that is, $x=y^*y-yy^*$ for some $y\in A$, if and only if it is the difference of two positive operators in A which are unitarily equivalent in A (cf. p.* 77[2] *Proposition* 6.2).

Proof. (→) By replacing y with $y+\lambda I$ for some real number λ we may assume that y is an invertible element, with the aid of the fact that

$$(y+\lambda I)^*(y+\lambda I)-(y+\lambda I)(y+\lambda I)^*$$

still remains to be x itself. We consider the left polar decomposition $y=up$ of y. Since u is a unitary operator (in A), $y^*y=pu^*up=p^2$, while $yy^*=up^2u^*$. Hence $x=p^2-up^2u^*$ with $p^2\geq 0$.

(←) Let $x=a-uau^*$, where a is a positive operator $a\in A$ and u is a unitary operator in A. We put $y=up$. Then, clearly, $x=y^*y-yy^*$, with $y\in A$.

From now on we only consider the case when A is an infinite semifinite factor acting on the Hilbert space H of arbitrary infinite dimension. Let rank (x) denote the relative rank of an element $x\in A$ as defined in (Definition 1 [7] p. 107). For the identity operator $I\in A$, we put $h=$ rank (I). For any cardinal number α with $\aleph_0\leq\alpha\leq h$, let J_α be the closed ideal in A by defining as the norm closure of the ideal $I_\alpha=\{x\in A:\text{rank}\,(x)<\alpha\}$ in A (p. 108, Definition 3 and Proposition 1 [7]). Let $q_\alpha: A\to A/J_\alpha$ be the quotient mapping and for $x\in A$, let $\sigma_\alpha(x)$ denote the spectrum of $q_\alpha(x)$, where $q_\alpha(x)$ is regarded as an element of A/J_α.

In what follows we describe a theory parallel to that developed in [2]. When the projection whose range is a closed subspace K of H belongs to A, we simply denote by $K\in A$ and call K belongs to A, without confusion.

LEMMA 3. *Let $x\in A$ and $\varepsilon>0$. Then there exists a closed subspace K of H, containing the kernal of A such that $K\in A$,*

$$\|xf\|<\varepsilon\|f\| \text{ for all } f\in K$$

such that $f\neq 0$, *and*

$$\|xf\|\geq\varepsilon\|f\| \text{ for all } f\notin K.$$

(In the presence of K with K=0, the first inquality is a vacuous statement.)
Furtheormore, if x is a normal operator (contained in A), there exists such a closed subspace which reduces x.

Proof. One can prove the lemma first for normal operator by using the spectral theorem, noticing that any range space of a spectral projection belongs to A. Then apply this to the positive part of the polar decomposition of an arbitrary $x\in A$ (cf. p. 63 Lemma 1.1 [2] and a remark preceeding to it).

LEMMA 4. *Let* $x\in A$ *and let* $\varepsilon>0$. *Suppose* K *is a closed subspace of* H *such that* $K\in A$, $\|xf\|<\varepsilon\|f\|$ *for all* $f\in K$ *with* $f\neq 0$, *and suppose* L *is a closed subspace of* H *such that* $L\in A$ *such that* $\|xf\|\geq\varepsilon\|f\|$ *for all* $f\in L^{\perp}$. *Then rank* $K\leq$ *rank* L *and rank* $L^{\perp}\leq$ *rank* $K^{\perp}$, *where rank* K *denote the relative rank of the projection in* A, *whose range is* K. *(Similarly, rank* L, *rank* $L^{\perp}$, *...etc will be understood from now on)* [7].

Proof. Let p, q denote respectively the projections in A whose range are L, K. If $f\in K$, $f\neq 0$, then $\|xf\|<\varepsilon\|f\|$ and hence $f\notin L^{\perp}$. It follows that $K\cap L^{\perp}=\{0\}$. Since the range projection of pq is $p-p\wedge(1-q)$, where q is the range of K (p. 119, Proposition 2.5.14 [4]), we see that rank (pq)=rank (qp) (p. 94, Theorem 4.3 [10])=rank $(q-q\wedge(1-p))$, since the range projection of qp is $q-q\wedge(1-p)$. But $q\wedge(1-p)=0$ since $q\wedge(1-p)$ is the projection whose range is $K\cap L^{\perp}$ $(=\{0\})$. Thus rank (pq)=rank q. It follows that rank K=rank q=rank $(pq)\leq$rank p=rank L.

Similarly let r, s denote respectively the projections in A whose ranges are $K^{\perp}$ and $L^{\perp}$. By the analagous argument as above we can easily prove that rank $L^{\perp}\leq$rank $K^{\perp}$.

REMARK 5. The subspace given in Lemma 3 is not unique. However it is an immediate result of Lemma 4 with $K=L$ that all the subspaces satisfying the conditions of Lemma 3 have the same relative rank whenever any one of them has an infinite relative rank and that they have all finite relative rank whenever any one of them has a finite relative rank (cf. p. 107, Definition 2 [7]). For the first case we put $\delta_\varepsilon(x)$ to be the common relative rank of all subspaces K satisfying the condition of Lemma 3, and for the second case we put $\delta_\varepsilon(x)=0$.

DEFINITION 6. We define the *relative approximate nullity* $\delta(x)$ of $x\in A$ to be

$$\delta(x)=\min_{\varepsilon>0}\ \delta_\varepsilon(x).$$

From here on, α will be an infinite cardinal with $\alpha\leq h$.

DEFINITION 7. A linear subspace K of H with $\bar{K}\in A$ is called *relatively α-closed* if there is a closed subspace L of H with $L\in A$ such that $L\subset K$ and such that rank $(\bar{K}\cap L^\perp)<\alpha$, where $\bar{K}$ denote the norm closure of K.

LEMMA 8. *If K and L are linear subspaces of H and K is closed while* rank$(\bar{L})<\alpha$, *with K, $L\in A$, then $K+L$ is relatively α-closed.*

Proof. $K+L\subset(K+\bar{L})^-$, $K+\bar{L}\subset(K+L)^-$, $(K+\bar{L})^-\subset(K+L)^-$, consequently, $(K+L)^-=(K+\bar{L})^-$. Thus $(K+L)^-\in A$, since K, $\bar{L}\in A$ and $(K+\bar{L})^-=K^\vee\bar{L}\in A$ (p. 119, Proposition 2. 5. 14). Now $K\subset K+L$ and rank$((K+L)^-\wedge K^\perp)=$rank$((K+\bar{L})^-\wedge K^\perp)=$rank$((K^\vee\bar{L})\wedge K^\perp)=$ rank$(\bar{L}-\bar{L}\wedge K)$ (p. 94, Corollary 4. 4 [10] the parallelogram rule) $\leq$ rank$(\bar{L})<\alpha$. By Definition 7, we see that $K+L$ is relatively α-closed.

LEMMA 9. *Let $x\in A$ be an operator whose restriction $K(\in A)$ is one-to-one. Let M be a linear subspace of K such that $\bar{M}\in A$ and that $x(M)$ is a closed subspace of H. Then*

$$\text{rank}\ \bar{M}^\perp\leq\text{rank}[x(M)]^\perp.$$

Proof. We note that $[x(M)]^-\in A$. The rest of the proof is similar to that of Lemma 4 and is omitted.

The proof of the next theorem can be done if we immitate the proof of Theorem 2. 6 [2] p. 64. There Lemmas 1. 1, 1. 2, 2. 4 and Theorem 0 in [2] were employed. In our situation we can apply Lemmas 3, 4, 8 in our article and Theorem 1 (p. 110 [7]). We omit the tedious execution of the proof.

THEOREM 10. *The following five conditions are equivalent for $x\in A$.*

(i) *x is left invertible in A modulo I_α.*
(ii) *$q_\alpha(x)$ is left invertible in A/J_α*
(iii) *x is bounded below on some closed subspace K belonging to A, with* $\mathrm{rank}(K^\perp)<\alpha$.
(iv) $\delta(x)<\alpha$. (v) $\mathrm{rank}(x)<\alpha$ *and $x(H)$ is relatively α-closed.*

DEFINITION 11. $x(\in A)$ is called a *relatively α-Fredholm operator* if $\mathrm{rank}(x)<\alpha$, rank $(H\ominus X(H)^-)$ $<\alpha$ and $x(H)$ is relatively α-closed.

THEOREM 12 (A general Atkinson's theorem). *Let $x\in A$. The following are equivalent.*
(i) *x is a relatively α-Fredholm operator.*
(ii) *x is an invertible element in A modulo I_α.*
(iii) *$q_\alpha(x)$ is an invertible element in A/J_α.*

Proof. This is proved in a similar way as that for Theorem 2.8 [2] p. 66. We apply Theorem 10 just as they did (Theorem 2.6 [2] p. 64) to prove Theorem 2.8 [2].

DEFINITION 13. Let $x\in A$. The *relative approximate point spectrum* of x, *of weight* α, denoted $\pi_\alpha(x)$, is the set of all complex numbers λ such that $\delta(x-\lambda I)\geq\alpha$. The *relative compression spectrum of x, of weight α,* denoted $\gamma_\alpha(x)$, is the set of all complex numbers λ such that rank $(H\ominus[(x-\lambda I)(H)]^-)\geq\alpha$. The *relative spectrum* of x, of *weight α,* denoted $\sigma_\alpha(x)$ is defined by $\sigma_\alpha(x)=\pi_\alpha(x)\cup\gamma_\alpha(x)$.

In what follows, we list a series of results analogous to those in [2], starting p. 67 in there. We shall omit the obvious proofs.

THEOREM 14. *Let $x\in A$ and $h=\mathrm{rank}(I)$, where I is the identity operator in A. Then the following conditions are equivalent.*
(i) $\lambda\in\pi_\alpha(x)$
(ii) *$q_\alpha(x)-\lambda q_\alpha(I)$ is not left invertible in A/J_α.*
(iii) *Every closed subspace K of H, with $K\in A$, on which $x-\lambda I$ is bounded below, has relative codimension $\geq\alpha$.*
(iv) *$x-\lambda I$ is not left invertible in A modulo I_α.*
(v) *Either λ is an eigenvalue of x of relative multiplicity at least α(i.e.,* rank *(null space $(x-\lambda I)\geq\alpha$), or $(x-\lambda I)(H)$ is not relatively α-closed.*

COROLLARY 15. *If $x\in J_\alpha$, then $\pi_\alpha(x)=\{0\}$.*

THEOREM 16. *Let* $x \in A$, $h = \mathrm{rank}(I)$. *The following are equivalent.*

(i) $\lambda \in \sigma_\alpha(x)$.

(ii) $q_\alpha(x) - \lambda q_\alpha(I)$ *is not invertible in* A/J_α.

(iii) $x - \lambda I$ *is not invertible in* A *modulo* I_α.

(iv) *Either* λ *is an eigenvalue of* x *of relative multiplicity at least* α, *or* rank $(H \ominus [(x-\lambda I)(H)]^-) \geq \alpha$, *or* $(x - \lambda I)(H)$ *is not relatively* α*-closed.*

COROLLARY 17. $\sigma_\alpha(x)$ *is precisely the ordinary spectrum of* $q_\alpha(x)$ *in* A/J_α. *Hence* $\sigma_\alpha(x)$ *is nonempty and compact.*

THEOREM 18. *Let* $x \in A$ ($\aleph_0 \leq \alpha \leq h$, *as always*). *Then* $\pi_\alpha(x)$ *contains the boundary of* $\sigma_\alpha(x)$.

Let x be a normal operator in $A(\subset L(H))$. By a version of the spectral theorem (cf. Section 97 of [3] and Section X.5 of [1]), there is an extended real valued positive measure μ defined on a σ-algebra on X and a unitary operator $U: H \longrightarrow L^2(X, \mu)$ such that UxU^* is the multiplication operator acting on $L^2(X, \mu)$ defined by a suitable essentially bounded measurable function ϕ.

THEOREM 19. *Let* $R_Y = U^*\{f \in L^2(X, \mu) : f$ *vanishes off* $\phi^{-1}(Y)\}$. *Then* $\lambda \in \sigma_\alpha(x)$ *if and only if* $\mathrm{rank}(p_Y) \geq \alpha$ *for every measurable neighborhood* Y *of* λ *where* $p_Y (\in A)$ *is the projection whose range is* R_Y.

LEMMA 20. *If* S *is a compact subset in the plane and* $\mathrm{rank}(p_S) \geq \alpha$ *then* $\pi_\alpha(x) \cap S \neq \phi$.

THEOREM 21. *Let* x *be a normal operator in* A. *Then* $x \in J_\alpha$ *if and only if* $\sigma_\alpha(x) = \{0\}$.

THEOREM 22. *Let* $x \in A$. *Then*

(i) $\pi_\alpha(x) = \pi_\alpha(x+y)$, *for any* $y \in J_\alpha$.

(ii) $\sigma_\alpha(x) = \sigma_\alpha(x+y)$, *for any* $y \in J_\alpha$.

THEOREM 23. *Let* x *be a normal operator in* A. *Then there is a normal operator* $y \in J_\alpha$ *such that* y *commutes with* x *and*

$$\sigma(x+y) = \sigma_\alpha(x),$$

where $\sigma(\cdot)$ *denote the usual spectrum of* $(\cdot)$ $((\cdot) \in L(H))$.

COROLLARY 24. *If* x *is a normal operator in* A, *then*

$$\sigma_\alpha(x) = \bigcap_{y \in J_\alpha} \sigma(x+y).$$

THEOREM 25. *Let* $y \in A$. *Then* $y \in J_\alpha$ *if and only if* $\sigma_\alpha(x) = \sigma_\alpha(x+y)$ *for all* $x \in A$.

Recall that an infinite cardinal α is called $\aleph_0$-irregular if it is the sum of countably many cardinals stricly smaller than α. A cardinal which is not $\aleph_0$-irregular is said to be $\aleph_0$-regular (p. 72 [2]).

DEFINITION 26. An operator $y \in A$ will be called *relatively α-Hilbert-Schmidt* operator if $H = \sum_{i=1}^{\infty} \oplus H_i$, $H_i \in A$, where $\operatorname{rank}([y(H_i)]^-) < \alpha$ for all $i = 1, 2, \ldots$, and $\sum_{i=1}^{\infty} \|y \mid H_i\|^2 < \infty$.

THEOREM 27. *Let* h $(=\operatorname{rank} I)$ *be an* $\aleph_0$*-irrdgular cardinal such that* $h > \aleph_0$. *Then every normal operator* $x (\in A)$ *can be written* $x = d + y$, *where* d *is a diagonal operator and* y *is a relatively* h*-Hilbert-Schmidt operator (and hence* $y \in J_h$).

THEOREM 28. *Let* h *be an* $\aleph_0$*-irregular cardinal with* $h > \aleph_0$ *Then any two normal operator* x *and* z *in* A *are unitarily equivalent modulo* J_h.

PROPOSITION 29. *Let* $x (\in A)$ *be a selfcommutator. Then* $\sigma_h(x)$ *containes at least one nonnegative real number and at least one nonpositive real number.*

3. The characterization of the selfcommutators in A.

LEMMA 30. *Let* x *be a normal operator in* A *and* $h = \operatorname{rank}(I)$, *where* I *is the identity operator in* A. *Then* x *may be decomposed,* $x = x_1 \oplus x_2$, *where each* x_i *acts on closed subspace* H_i *with* $H_i \in A$ *such that the identity operator* I_i *on* H_i *has the relative* rank h *with respect to* A_i *(Here* A_i *denotes the reduced von Neumann algebra with respect to* $H_i \in A$), *such that*

$$\|x_2 - \lambda\| \leq \frac{1}{2} \|x_1 - \lambda\| \text{ and } \lambda \in \sigma_h(x_2).$$

Proof. This can be proven by the multiplcation operator version of the spectral theorem. We omit the detail.

LEMMA 31. *Let x be a normal element in A. Let $\lambda \in \sigma_h(x)$. Then x can be written as $x = \sum_{n=0}^{\infty} \oplus x_n$ such that each x_n acts on a closed subspace H_i with $H_i \in A$,* $\operatorname{rank}(H_i) = h$, *and $\|x_n - \lambda\| \leq 2^{-n} \|x - \lambda\|$, for all $n = 0, 1, 2, \ldots$.*

LEMMA 32. *Suppose x is a hermitian element in A of the form $x = \sum_{i=0}^{\infty} \oplus x_i$ such that each x_i acts on a closed subspace H_i with $H_i \in A$,* $\operatorname{rank}(H_i) = h$, *for all $i = 0, 1, 2, \ldots$. We thus can regared each x_i acts on H itself. Define the operators y_n, z_n on H, for $n = 0, 1, 2, \ldots$, by*

$$y_n = \sum_{i=0}^{n} x_{2i} \text{ and } z_n = -\sum_{i=0}^{n} x_{2i+1}.$$

If there is bound r such that $\|y_n\| < r$ and $\|z_n\| < r$ for all $n = 0, 1, 2, \ldots$ then x is a selfcommutator in A.

Proof. Define p on H by $p = \sum_{i=0}^{\infty} \oplus p_i$, where

$$p_1 = 0, \ p_{2i} = y_i, \ i = 0, 1, 2, 3, \ldots,$$
$$p_{2i+1} = z_{i-1}, \ i = 1, 2, 3, \ldots.$$

similarly, define $q = \sum_{i=0}^{\infty} \oplus q_i$ where $q_0 = 0$, $q_{2i} = y_{i-1}$, $i = 1, 2, 3, \ldots$, and $q_{2i+1} = z_i$, $i = 0, 1, 2, \ldots$. Then p and q are bounded hermitian operators in A. In fact $\|p\| \leq r$ and $\|q\| \leq r$. What it remains to verify is that p and q are unitarily equivalent via a unitary element u in A. But this will be clearified by the next lemma, whose is omitted.

LEMMA 33. *Let $y \in A$ and $\{e_n\}_{n=1}^{\infty}$ be an orthogonal family of projections in A such that $\sum_{n=1}^{\infty} e_n = I$ with respect to the strong operator topology. Assume that each $H_i = e_i(H)$ reduces y. We put $y_i = y \mid H_i$. Let σ be a permutation of $\{1, 2, 3, \ldots\}$, that is, it is a one-to-one function of the set of all natural numbers. Let us consider $y_\sigma = \sum_{i=1}^{\infty} \oplus y_{\sigma(i)}$. Here, $y_{\sigma(i)}$ is defined as the operator acting on $H_{\sigma(i)}$, given by $u_{\sigma(i)} y_i u_{\sigma(i)}^{-1}$, where $u_{\sigma(i)} = v_{\sigma(i)} \mid H_i$, $v_{\sigma(i)} \in A$, $v_{\sigma(i)} v_{\sigma(i)}^* = e_{\sigma(i)}$, $v_{\sigma(i)}^* v_{\sigma(i)} = e_i$. Define $u : H \longrightarrow H$ by $u \mid H_i = u_{\sigma(i)}$, $i = 1, 2, 3, \ldots$. Then u is a unitary operator on H such that $u \in A$ and $uyu^* = y_\sigma$.*

PROPOSITION 34. *If x is a hermitian operator in A and $0 \in \sigma_h(x)$, then x is a selfcommutator.*

Proof. Decompose x according to Lemma 31 and then apply Lemma 32.

COROLLARY 35. *If $x(\in A)$ is a hemitian operator with* rank *(null space of x) $\geq h$, then x is a selfcommutator.*

COROLLARY 36. *If $x(\in A)$ is hemitian operator and $x \in J_h$, then it is a selfcommutator in A.*

PROPOSITION 37. *Let $x(\in A)$ be a hermitian operator. If $\sigma_h(x)$ has both a positive and a negative real number, then x is a selfcommutator.*

Proof. On can argue just as the proof of Proposition 6.8 (p. 78 [2]). We omit the details.

THEOREM 38. *Let A be an infinite semifinite factor on a Hilbert space and I be the identity operator of A. Let $x(\in A)$ a selfadjoint element of A. Then x is a selfcommutator in A if and only if $\sigma_h(x)$ contains at least one nonnegative and at least one nonpositive real number.*

Proof. We simply combine Propositions 29, 34 and 37.

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