J. Korean Math. Soc. 23(1986), No. 1, pp. 67~72

FINITENESS OF INTEGRAL DIFFERENTIAL FORMS

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1. Preliminaries

Let A be a commutative ring with unit, and M an A-module. Suppose that $T(M)$ is a tensor algebra of M and $T(M)^*$ the dual module of $T(M)$. Let

$$K_n(M) = \{f : f \in T(M)^*,\ f \mid T_k(M) = 0 \text{ for all } k \neq n\},$$

and

$$K(M) = \sum_{n=0}^{\infty} K_n(M).$$

Then $K(M)$ is a submodule of $T(M)^*$ and the sum is direct. $K(M)$ can be made into an algebra by defining multiplication as follows: Let N and P be A-modules, and N^* and P^* dual modules. For $f \in N^*$, $g \in P^*$, define $f*g \in (N \otimes P)^*$ by $(f*g)(a \otimes c) = f(a)g(b)$, $a \in N$, $b \in P$. To define a multiplication in $K(M)$, let

$$\gamma_{n,m} : T_n(M) \otimes T_m(M) \longrightarrow T_{n+m}(M)$$

be the canonical isomorphism, i. e., $x \otimes y \longrightarrow xy$, $x \in T_n(M)$, $y \in T_m(M)$. Let $\eta_{n,m} = \gamma_{n,m}^{-1}$, and $\eta_{n,m}^*$ be the dual homomorphism. For f, $g \in K(M)$, we define a product by

$$fg = \sum_{n,m} \eta_{n,m}^* ((f \circ j_n) * (g \circ j_m)) \circ p_{n+m},$$

where $j_n : T_n(M) \longrightarrow T(M)$, $j_m : T_m(M) \longrightarrow T(M)$ are natural injections and $p_{n+m} : T(M) \longrightarrow T_{n+m}(M)$ is the $(n+m)$ th projection. $K(M)$ is called the algebra of multilinear forms on M.

Let R be an integral domain, K a field containing R. $\mathcal{D}_{K/R}$ will denote the K-module of all R-derivations $D : K \longrightarrow K$. The algebra of multilinear forms on $\mathcal{D}_{K/R}$ is called the *algebra of differential forms*

Received October 5, 1985.

(on K), and will be denoted by $D(K/R)$, and the submodule of homogeneous elements of degree k by $D_k(K/R)$. The elements of $D(K/R)$ are called differential forms (on K). Let $d: K \longrightarrow D_1(K/R)$ be an R-derivation defined by $d(a)(D)=D(a)$ for $D\in\mathfrak{D}_{K/R}(=T_1(\mathfrak{D}_{K/R}))$ and $d(a)\,|\,T_k(\mathfrak{D}_{K/R})=0$ for $n\neq 1$. A differential form $x\in D(K/R)$ will be said to be integral if $x\in\sum_k S(dS)^k$ for all valuation rings S in K containing R. Here $S(dS)^k=\{\sum s_0 s d_1 \ldots ds_k \,|\, s_i\in S\}$. Integral differential forms in $D_k(K/R)$ are called ***homogeneous differential forms of degree k.***

2. Finiteness

In this section, R will denote a Noetherian integrally closed domain; $P=R[x_1, \ldots, x_n]$ a polynomial ring over R in $x_1, \ldots, x_n$; K_0 a field of quotients of P; and K a finite separable algebraic extension of K_0. It is well known that $D_1(K/R)$, in this case, is a vector space over K with $\{dx_1, \ldots, dx_n\}$ as basis, where $d: K\longrightarrow D_1(K/R)$ is an R-derivation as is defined in Section 1; and $D(K/R)$ is isomorphic to a tensor algebra of $D_1(K/R)$. Hence a homogeneous differential form of degree k is uniquely expressed in the form

$$x=\sum_i a_i dx_{i_1}\ldots dx_{i_k},\quad a_i\in K,\quad 1\leq i_1, \ldots, i_k\leq n.$$

The main result in this section is that the R-module of homogeneous differential forms of degree k is finitely generated.

Since K is a finite separable algebraic extension of K_0, let $K=K_0(\alpha)$ and $f(X)=X^m+a_1X^{m-1}+\cdots+a_m$, $a_i\in K_0$, be the minimal polynomial of α. Suppose that S_0 is a discrete rank one valuation ring in K_0 containing P, and let $\mathcal{A}$ be the set of all valuation rings in K which are extensions of S_0. $\bar{S}_0$ will denote the integral closure of S_0 in K.

LEMMA 1. *If* $\alpha\in\bar{S}_0$,

$$(f'(\alpha))^{3k+1}\Big(\bigcap_{S\in\mathcal{A}} S(dS)^k\Big)\subseteq\sum_i S_0[\alpha]dx_{i_1}\ldots dx_{i_k},$$

where $1\leq i_1, \ldots, i_k\leq n$.

Proof. $\alpha\in\bar{S}_0$ implies that $f(X)\in S_0[X]$. Since $f(\alpha)=0$, and $a_i\in S_0$ which is a localization of P,

$$0=d(f(\alpha))=f'(\alpha)d\alpha+\sum_{i=1}^{m-1}\alpha^{m-i}d(a_i),$$

where $\alpha^{m-i}d(a_i) \in \sum_{i=1}^{m} \bar{S}_0\ dx_i$.

It follows that

(1) $$f'(\alpha)d\alpha \in \sum_{i=1}^{m} \bar{S}_0 dx_i.$$

It is well known that (c. f. [3])

(2) $$f'(\alpha)\bar{S}_0 \subseteq S_0[\alpha]$$

Hence, for any $b \in \bar{S}_0$,

$$d(f'(\alpha)b) \in d(S_0[\alpha]) \subseteq \sum_{i=1}^{m} S_0[\alpha]dx_i + S_0[\alpha]d\alpha.$$

On the other hand,

$$d(f'(\alpha)b) = f'(\alpha)db + bd(f'(\alpha)),$$

where $bd(f'(\alpha)) \in \sum_{i=1}^{m} \bar{S}_0 dx_i + \bar{S}_0 d\alpha$.

Hence, $f'(\alpha)db \in \sum_{i=1}^{m} \bar{S}_0 dx_i + \bar{S}_0 d\alpha$.

By using (2), it follows that

$$(f'(\alpha))^2 d\bar{S}_0 \subseteq \sum_{i=1}^{m} S_0[\alpha]dx_i + S_0[\alpha]d\alpha.$$

Also,

$$(f'(\alpha))^{2k}(d\bar{S}_0)^k \subseteq \sum_i S_0[\alpha]dx_{i_1}\dots dx_{i_j}(d\alpha)^{k-j},$$

and by using (1),

$$f'(\alpha)^{3k}(d\bar{S}_0)^k \subseteq \sum_i S_0[\alpha]dx_{i_1}\dots dx_{i_k}.$$

It is will known that S is a localization of $\bar{S}_0$, and therefore,

$$SdS \subseteq Sd\bar{S}_0$$

Hence,

$$\begin{aligned}(f'(\alpha))^{3k}\left(\bigcap_{S\in A} S(dS)^k\right) &= (f'(\alpha))^{3k}\left(\bigcap_{S\in A} S(d\bar{S}_0)^k\right)\\ &\subseteq \sum_i \left(\bigcap_{S\in A} S\right)dx_{i_1}\dots dx_{i_k}\\ &= \sum_i \bar{S}_0 dx_{i_1}\dots dx_{i_k}.\end{aligned}$$

By multiplying $f'(\alpha)$ and using (2) again,

$$(f'(\alpha))^{3k+1}(\bigcap_{S\in\mathcal{A}} S(dS)^k) \subseteq \sum_i S_0[\alpha]dx_{i_1}\dots dx_{i_k}.$$

LEMMA 2. *Let I_k be the R–module of integral differential forms degree k, and $\bar{P}$ the integral closure of P in K. If $\alpha\in\bar{P}$,*

$$(f'(\alpha))^{3k+1}I_k \subseteq \sum_i P[\alpha]dx_{i_1}\dots dx_{i_k}.$$

Proof. Let $\mathcal{B}_0$ be the set of all discrete rank one valuation rings containing P, and $\mathcal{B}$ the set of all extension valuation rings of members of $\mathcal{B}_0$ in K. Since P is Noetherian integrally closed domain, $P=\bigcap_{S_0\in\mathcal{B}} S_0$, and it follows that

$$\begin{aligned}(f'(\alpha))^{3k+1}I_k &\subseteq (f'(\alpha))^{3k+1}(\bigcap_{S\in\mathcal{B}} S(dS)^k)\\ &\subseteq \sum_i \bigcap_{S_0\in\mathcal{B}_0} S_0[\alpha]dx_{i_1}\dots dx_{i_k} \text{ (by lemma 1)}\\ &= \sum_i P[\alpha]dx_{i_1}\dots dx_{i_k}.\end{aligned}$$

LEMMA 3. *Under the same assumption as lemma 7, in fact, there exists a natural number r such that*

$$(f'(\alpha))^{3k+1}I_k \subseteq \sum_i M dx_{i_1}\dots dx_{i_k},$$

where $M=\sum_{s_i\leq r} Rx_1^{s_1}\dots x_n^{s_n}z^{s_{n+1}}$.

Proof. Consider the subring $P_1=R[x_1^{-1}, x_2, \dots, x_n]$ of K_0. Let S_1 be the localization of P_1 at the prime ideal (x_1^{-1}) of P_1. Put $y_1=x_i^{-1}$ and $y_i=x_i$ for $i=2, \dots, n$. Then K_0 is the field of quotients of $P_1=R[y_1, \cdots y_n]$.

Let $\alpha_1=x_1^{-h}\alpha$ for some positive integer h, and as before, let

$$f(X)=X^m+a_1X^{m-1}+\cdots+a_m,\ a_i\in K_0,$$

be the minimal polynomial of α over K_0. Then

$$0=f(\alpha)=x_1^{hm}g(\alpha_1),$$

where $g(X)=X^m+a_1x_1^{-h}X^{m-1}+\cdots+a_iX_1^{-hi}X^{m-i}+\cdots+a_mx_1^{-hm}$. Actually, g is the minimal polynomial of α_1 over K_0, and $K=K_0(\alpha_1)$. Moreover, we can put h sufficiently large so that all coefficients $a_ix_1^{-hi}$, $i=1, \dots, m$, of g are contained in S_1. Then $\alpha_1\in\bar{S}_1$. Let $\mathcal{A}_1$ be the set of all extensions of members of S_1 in K. By using lemma 1,

$$(1)\qquad (g'(\alpha_1))^{3k+1}(\bigcap_{S\in\mathcal{A}_1} S(dS)^k) \subseteq \sum_i S_1[\alpha_1]dy_{i_1}\dots dy_{i_k}$$

$$\subseteq \sum_i S_1[\alpha_1](-x_1^{-2})^{q_i} dx_{i_1} \dots dx_{i_k},$$

(where q_i is the number of y_1 among $y_{i_1} \dots y_{i_k}$)

$$\subseteq \sum_i S_1[\alpha] dx_{i_1} \dots dx_{i_k},$$

since $S_1[\alpha_1](x_1^{-2})^{q_i} \subseteq S_1[\alpha]$.
We have,

$$(f'(\alpha))^{3k+1} I_k \subseteq (f'(\alpha))^{3k+1} (\bigcap_{S \in \mathfrak{A}_1} S(dS)^k)$$
$$= x_1^{h(m-1)(3k+1)} (g'(\alpha_1))^{3k+1} (\bigcap_{S \in \mathfrak{A}_1} S(dS)^k)$$
$$x_1^{h(m-1)(3k+1)} \sum_i S_1[\alpha] dx_{i_1} \dots dx_{i_k}, \text{ by } (1).$$

Let $r_1 = h(m-1)(3k+1)$. Then

(2) $$(f'(\alpha))^{3k+1} I_k \subseteq \sum_i x_1^{r_1} S_1[\alpha] dx_{i_1} \dots dx_{i_k}.$$

On the other hand, by lemma 2,

(3) $$(f'(\alpha))^{3k+1} I_k \subseteq \sum_i R[x_1, \dots, x_n, \alpha] dx_{i_1} \dots dx_{i_k}.$$

Combining (2) and (3), we obtain

(4) $$(f'(\alpha))^{3k+1} I_k \subseteq \sum_i \sum_{s_1 \leq n_1} R[x_2, \dots, x_n, \alpha] x^{s_1} dx_{i_1} \dots dx_{i_k}.$$

Similarly, it can be shown that for each $j = 2, \dots, n$, there exists r_j such that

(5) $$(f'(\alpha))^{3k+1} I_k \subseteq \sum_i \sum_{s_j \leq r_j} R[x_1, \dots, \hat{x}_i, \dots, x_n z] x_j^{s_j} dx_{i_1} \dots \cdots dx_{i_k},$$

where $\hat{x}_j$ denotes the omission of x_j.
Also, since α is algebraic over K_0 and $\alpha \in \bar{P}$,

$$R[x_1, \dots, x_n, \alpha] = \sum_{s_{n+1} \leq \deg f} R[x_1, \dots, x_n] \alpha^{s_{n+1}},$$

Hence,

(6) $$(f'(\alpha))^{3k+1} I_k \subseteq \sum_{s_{n+1} \leq \deg f} R[x_1, \dots, x_n] \alpha^{s_{n+1}} dx_{i_1} \dots \cdots dx_{i_k}.$$

Let $r = \max\{r_1, \dots, r_n, \deg f\}$. Then it follows from (4), (5), and (6) that

$$(f'(\alpha))^{3k+1} I_k \subseteq \sum_i M dx_{i_1} \dots dx_{i_k}$$

THEOREM 1. *Let R be an integrally closed Noetherian domain, and K*

a finitely and separably generated extension field over the field of quotients of R. Then the module of homogeneous integral differential forms on K of degree n is a finitely generated R-module.

Proof. Since K is finitely and separably generated over the field of quotients of R, there exist $x_1, \ldots, x_n$, $\alpha \in K$ such that $P = R[x_1, \ldots, x_n]$, $K = K_0(\alpha)$ where K_0 is the field of quotients of P, and α is separable algebraic over K_0. Without loss of generality, we can assume that the minimal polynomial $f(X) \in P[X]$. In this case, $\alpha \in \bar{P}$, · the integral closure of P in K. We have all assumptions for lemma 2, and hence by lemma 3,

$$I_k \subseteq \sum_i M(f'(\alpha))^{-(3k+1)} dx_{i_1} \ldots dx_{i_k},$$

Where $M = \sum_{s_i \leq r} R x_1^{s_1} \cdots x_n^{s_n} \alpha^{s_{n+1}}$.

It follows that I_k is a submodule of a finitely generated R-module. Since R is Noetherian, I_k itself is a finitely generated R-module.

COROLLARY 1. *Let R be an integrally closed Notherian domain, and K a finitely and separably generated extension field over the field of quotients of R. Then the integral closure of R in K is a finitely generated R-module.*

Proof. Integral differential forms of degree zero are elements of K which are integrally dependent on R. Hence, this is a special case of THEOREM 1.

References

1. I. Y. Chung, *Free Joins of Algebras and Kähler's Differential Forms*, Abh. Math. Sem. Univ. Hamburg **35** (1970), 92–96.
2. E. Kähler, *Algebra und Differentialrechnung*, bericht uber die Mathematiker-Tagung in Berlin, Januar 1953, 58–163.
3. O. Zariski and P. Samuel, *Commutative algebra*, Vol. **I**, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960.

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