J. Korean Math. Soc. 23(1986), No. 1, pp. 19~33

SOME REMARKS ON CONSERVATIVE FUNCTORS WITH LEFT ADJOINTS

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1. Introduction

A functor $T:\mathscr{A}\to\mathscr{B}$ is said to be *conservative* if it reflects isomorphisms: that is, any morphism f of $\mathscr{A}$ with Tf invertible is itself invertible. We are interested here in those functors which, like the forgetful functors of algebra, are conservative and have left adjoints. Such functors "nearly" enjoy a variety of good properties, in the sense that they do so under more-or-less mild completeness or cocompleteness conditions on $\mathscr{A}$. Our aim in this note is to give such conditions on $\mathscr{A}$ which are as weak as we can make them.

Let the adjunction be $\eta, \varepsilon : S \dashv T : \mathscr{A}\to\mathscr{B}$. As is well known, each $\varepsilon_A : STA\to A$ is an epimorphism if and only if T is faithful, while each ε_A is invertible if and only if T is fully faithful. The intermediate hypothesis that each ε_A is a strong epimorphism implies that T is conservative; but the converse needs some conditions on $\mathscr{A}$.

Since a T with a left adjoint preserves limits, a conservative one also *reflects* $\mathscr{K}$-indexed limits if $\mathscr{A}$ admits such limits. Without the hypothesis that $\mathscr{A}$ admits these limits, however, a proof that T reflects them seems to need each ε_A to be, not only a strong epimorphism, but in fact a *familially strong* epimorphism in the sense of [4]. Recall from [4] that regular epimorphisms are familially strong; we give below various conditions on $\mathscr{A}$ which ensure that all strong epimorphisms are familially strong.

The first author acknowledges the support of a grant from the Republic of Korea which permitted a year-long visit to Sydney in 1983/84, and the second author acknowledges the support of the Australian Research Grants Scheme.

Received February 25, 1985.

Given the adjunction $S \dashv T$ as above, and given $P : \mathscr{C} \to \mathscr{A}$ such that $TP : \mathscr{C} \to \mathscr{B}$ has a left adjoint, the classical adjoint-triangle theorems of Dubuc [1] and Huq [3] assert that P has a left adjoint if each ε_A is a coequalizer and $\mathscr{C}$ admits coequalizers. There should, however, be adjoint-triangle theorems asserting that P has a left adjoint if T and TP do, provided that T is conservative-given some conditions on $\mathscr{A}$ to ensure the good behaviour of T, and appropriate completeness conditions on $\mathscr{C}$; for then P preserves limits, since TP does so and T reflects them. One such result can be deduced from the adjoint-triangle theorem of Tholen [7] and the results foreshadowed above: but we give others too which use cocompleteness rather than completeness properties of $\mathscr{C}$.

2. Classes of epimorphisms

Following [5], we call a map $p : A \to B$ in a category $\mathscr{A}$ a ***regular epimorphism*** if it is the joint coequalizer of some family (not necessarily small) of pairs $x_i, y_i : C_i \to A$. It comes to the same thing to say that p is the joint coequalizer of the family of *all* pairs $x, y : C_{xy} \to A$ satisfying $px = py$. We call p a ***coequalizer*** if it is the coequalizer of a single pair $x, y : C \to A$; thus every coequalizer, and in particular every retraction, is a regular epimorphism. On the other hand, if $\mathscr{A}$ admits pullbacks, every regular epimorphism is a coequalizer: namely, of the pair x, y obtained by pulling back p along itself. Yet for a general $\mathscr{A}$, these classes differ.

We recall from [4] that the epimorphism $p : A \to B$ is said to be ***familially strong*** if, whenever $(m_i : X \to Y_i)_{i \in I}$ is a jointly-monomorphic family of maps, and whenever $u : A \to X$, $v_i : B \to Y_i$ are such that $m_i u = v_i p$ for each i, there is a (necessarily unique) $w : B \to X$ with $wp = u$ and $m_i w = v_i$ for each i. To say that p is ***small-familially-strong*** is to require this only for ***small*** families (m_i); while to say that p is ***strong*** (see [5]) is to require it only when $I = 1$, so that the family (m_i) reduces to a single monomorphism $m : X \to Y$. We showed in Proposition 4.4 of [4] that regular epimorphisms are familially strong, which goes beyond the older result of [5] that they are strong.

An epimorphism $p : A \to B$ is often said to be *extremal* if it factorizes through no proper subobject of B; that is to say, whenever $p = mu$ with m monomorphic, m is invertible. Clearly every strong epimorphism is extremal; while the converse is true (see [5]) if $\mathscr{A}$ admits pullbacks, as well as under some other conditions we examine in the next section, although not in general. Of course, any extremal epimorphism–and in particular any strong one–that is also a monomorphism is necessarily invertible.

The class of extremal epimorphisms has in a general $\mathscr{A}$ no good closure properties to speak of, while that of coequalizers has but few: which is why these classes have little importance, except as names for the properties they signify. We saw in [4], however, that the classes of regular epimorphisms, of familially–strong epimorphisms, of small–familially–strong epimorphisms, of strong epimorphisms, and of (mere) epimorphisms, are each closed under colimits; so that, since each contains the identities, each enjoys all the other closure properties of Theorem 2.5 of [4]. In addition, each of these classes except that of regular epimorphisms is closed under composition. Moreover, each of these classes is preserved by any functor S with a right adjoint, since S preserves colimits and its right adjoint T preserves jointly–monomorphic families.

As an isomorphism class of monomorphisms [resp. strong monomorphisms, regular monomorphisms] with codomain A is commonly called a ***subobject*** [resp. ***strong subobject***, ***regular subobject***] of $\mathscr{A}$, so an isomorphism class of epimorphisms [resp. strong epimorphisms, regular epimorphisms] with domain A may be called a ***quotient object*** [resp. ***strong quotient object***, ***regular quotient object***] of A. The category $\mathscr{A}$ is ***wellpowered*** [resp. ***weakly wellpowered***] if each object has but a small set of subobjects [resp. strong subobjects]; the duals are ***cowellpowered*** and ***weakly cowellpowered***. We remind the reader that most of the usual categories of structures one meets in practice are wellpowered and cowellpowered. An exception is Spanier's category [6] of quasi–topological spaces. This is not wellpowered, but is weakly so; and even though it is not wellpowered, it admits ***arbitrary*** intersections–that is, even large ones–of subobjects. The same statements are true of its dual, and also of the category $\infty+1$ of ordinals not exceeding the first inaccessible one, and of its dual. Even the category of ***finite*** sets admits ***arbitrary*** intersections of subobjects, as does its dual. Thus neither weak wellpoweredness, nor the existence of arbitrary intersections, is unreasonably strong as a completeness hypothesis. In future, chiefly because some authors use "quotient" in a different sense from the above, we shall speak of *intersections of monomorphisms* and *cointersections of epimorphisms*, rather than "intersections of subobjects" and "cointersections of quotient objects".

We now examine conditions under which all strong epimorphisms are familially strong, or at least small-familially-strong. The first two involve completeness conditions on $\mathscr{A}$, and the other two involve cocompleteness conditions.

PROPOSITION 2.1. *If $\mathscr{A}$ admits small products, every strong epimorphism is small-familially-strong.*

Proof. The familially-strong condition with respect to the small jointly-monomorphic family $(m_i : X \to Y_i)$ reduces to that with respect to the single monomorphism $m : X \to \prod_i Y_i$.

LEMMA 2.2. *Consider families of maps $(m_i : X \to Y_i)_{i \in I}$ and $(v_i : B \to Y_i)_{i \in I}$. If A admits non-empty finite limits and arbitrary intersections of regular monomorphisms, the diagram*

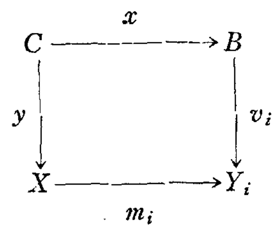
$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle v_i} \\ X & \xrightarrow[m_i]{} & Y_i \end{array} , \qquad (2.1)$$

wherein i takes all values in I, admits a limit.

Proof. Form for each i the pullback

$$\begin{array}{ccc} C_i & \xrightarrow{x_i} & B \\ {\scriptstyle y_i}\downarrow & & \downarrow{\scriptstyle v_i} \\ X & \xrightarrow[m_i]{} & Y_i \end{array} ,$$

and observe that $(x_i, y_i) : C_i \to B \times X$ is a regular monomorphism, being the equalizer of $v_i r, m_i s : B \times X \to Y_i$ where r, s are the projections. If $(x, y) : C \to B \times X$ is the intersection of these regular monomorphisms, then

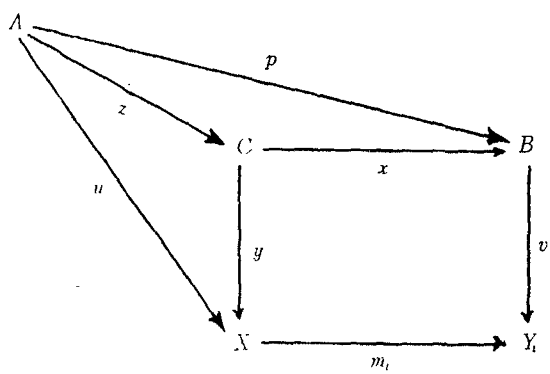


(2. 2)

is the limit of (2. 1).

PROPOSITION 2. 3. *If $\mathscr{A}$ admits non-empty finite limits and arbitrary intersections of regular monomorphisms, every strong epimorphism is familially strong.*

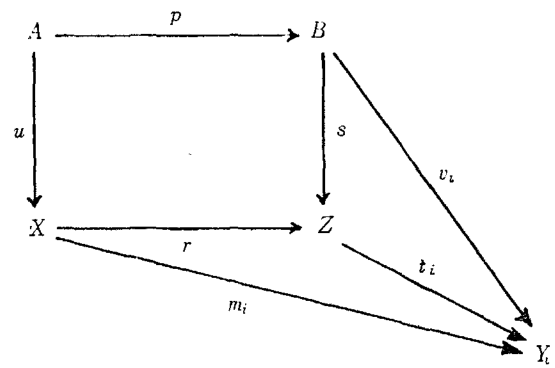
Proof. Given the situation in the definition above of "familially strong", form the limit (2. 2) and define z by the commutativity of



Then x is monomorphic; for $xg=xh$ gives $v_ixg=v_ixh$ and so $m_iyg=m_iyh$, whence $yg=yh$ since (m_i) is jointly monomorphic, so that $g=h$. When p is a strong epimorphism it follows that x is invertible, and then $w=yx^{-1}$ gives the desired "diagonal".

PROPOSITION 2. 4. *If $\mathscr{A}$ admits pushouts, every strong epimorphism is familially strong.*

Proof. Given the situation in the definition above of "familially strong", form the pushout Z of p and u, and define t_i by the commutativity of



When p is a strong epimorphism, so is its pushout r, by [5] or [4]; but r is also monomorphic since (m_i) is jointly monomorphic, so that r is invertible and $w=r^{-1}s$ provides the desired "diagonal".

PROPOSITION 2.5. *If $\mathscr{A}$ admits coequalizers and arbitrary cointersections of strong epimorphisms, every strong epimorphism is familially strong.*

Proof. Given a strong epimorphism $p : A \to B$, let $u : A \to C$ be the cointersection of all the familially-strong epimorphisms through which p factorizes; then $p=mu$ for some m, and u is itself a familially-strong epimorphism by Theorem 2.5 of [4]. Now m is monomorphic; for if $mx=my$, we have $m=nz$ where z is the coequalizer of x and y, and p factorizes through the familially-strong epimorphism zu, so that z is invertible by the definition of u, and $x=y$. Thus m is invertible since p is a strong epimorphism, so that p like u is familially-strong.

We conclude that, in "good" categories, the only distinct classes of epimorphisms among those we have named above are the retractions, the regular epimorphisms, the strong epimorphisms, and the epimorphisms. These are truly distinct in general, for instance in the essentially-algebraic category **Cat** of small categories. For remarks on coincidences among these in special cases, see [5].

3. Factorizations into a strong epimorphism and a monomorphism

Write *Epi* and *Mon* for the classes of epimorphisms and monomorphisms in $\mathscr{A}$, and *SEpi*, *SMon* for the classes of strong ones. If every map $f : A \to B$ admits a factorization $f = me$ with $m \in Mon$ and $e \in SEpi$, we say that *(SEpi, Mon) factorizations exist*; in this case, $(SEpi, Mon)$ is a *factorization system* in the sense of [2].

We can regard the existence of such factorizations as itself being a kind of "completeness" condition. Of course it may well hold in categories subjected to no infinite completeness or cocompleteness conditions: as it does in abelian categories or elementary topoi. Again, if $\mathscr{A}$ admits pullbacks and coequalizers, we can form maps $x, y : C \to A$ by pulling back f along itself, take e to be the coequalizer of these, and set $f = me$. Then m is a monomorphism if it *happens* that regular epimorphisms are closed under composition in $\mathscr{A}$, which is equally to say that every strong epimorphism in $\mathscr{A}$ is regular; see [5]. In general, however, this is false: and to get existence results for $(SEpi, Mon)$ factorizations that apply to all "reasonably good" $\mathscr{A}$, we have to assume that $\mathscr{A}$ admits arbitrary intersections of monomorphisms, or arbitrary cointersections of strong epimorphisms. We now turn to such results.

PROPOSITION 3.1. *Consider the following assertions:*

(i) *(SEpi, Mon) factorizations exist.*

(ii) *If $f : A \to B$ factorizes through no proper subobject of B, it is a strong epimorphism.*

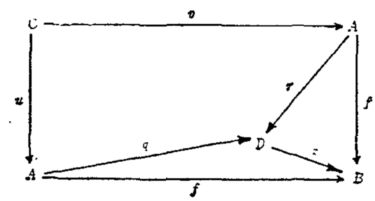
(iii) *If $f : A \to B$ factorizes through no proper strong epimorphism $q : A \to D$, it is a monomorphism.*

Then (i) *implies* (ii) *and* (iii), *while* (ii) *implies* (i) *if $\mathscr{A}$ admits arbitrary intersections of monomorphisms, and* (iii) *implies* (i) *if $\mathscr{A}$ admits arbitrary cointersections of strong epimorphisms.*

Proof. That (i) implies (ii) and (iii) is trivial. For the converses, we factorize f as me, where for (ii) m is the intersection of the monomorphisms $j : E \to B$ through which f factorizes, while for (iii) e is the cointersection of the strong epimorphisms $p : A \to E$ through which f factorizes (and is hence itself a strong epimorphism by [5] or [4]).

PROPOSITION 3.2. *An $f : A \to B$ that factorizes through no proper strong epimorphism $q : A \to D$ is a monomorphism if $\mathscr{A}$ admits coequalizers, or if $\mathscr{A}$ admits both pushouts and pullbacks.*

Proof. If $\mathscr{A}$ admits coequalizers and $fx = fy$, then $f = zq$ where q is the coequalizer of x, y; so that q is invertible and $x=y$. If $\mathscr{A}$ admits both pushouts and pullbacks, let the outer square below be the pullback of f by itself and let the inner square be the pushout of u and v:



Since $f1_A=f1_A$, there is a $w : A\to C$ with $uw=vw=1$. Thus v, as a retraction, is a strong epimorphism; whence by [5] or [4] its pushout q is a strong epimorphism. So q is invertible by hypothesis. Multiplying the equation $q^{-1}rv=u$ on the right by w gives $q^{-1}r=1$; whence $v=u$, so that f is monomorphic.

PROPOSITION 3.3. *An $f : A\to B$ that factorizes through no proper subobject of B is a strong epimorphism if $\mathscr{A}$ admits pullbacks and admits either equalizers or pushouts.*

Proof. In either case, f is an epimorphism by Proposition 3.2, and hence an extremal epimorphism. We now use the very simple result of [5] that extremal epimorphisms are strong if $\mathscr{A}$ admits pullbacks.

Omitting the "mixed" case where $\mathscr{A}$ admits both pullbacks and pushouts, so that our conditions refer to limits or to colimits but not to both, and combining the results above with those of Section 2, we have:

THEOREM 3.4. *$\mathscr{A}$ admits (SEpi, Mon) factorizations if it admits equalizers, pullbacks, and arbitrary intersections of monomorphisms; if it also admits binary products, all the strong epimorphisms are familially strong. Again, $\mathscr{A}$ admits (SEpi, Mon) factorizations if it admits coequalizers and arbitrary cointersections of strong epimorphisms, and then all the strong epimorphisms are familially strong.*

We need below the simple generalization of the latter case to factorizations of families:

PROPOSITION 3.5. *Let $\mathscr{A}$ admit coequalizers and arbitrary cointersections of strong epimorphisms. Then any family $(f_i : A\to B_i)$ factorizes as $f_i=m_ie$, where e is a strong epimorphism (in fact, familially strong)* *and* (m_i) *is a jointly-monomorphic family.*

Proof. Now e is the cointersection of those strong epimorphisms $p : A \to E$ through which *every* f_i factorizes; and that (m_i) is jointly monomorphic follows as in the proof of Proposition 3.2 from the existence of coequalizers.

4. Conservative functors with left adjoints

We consider an adjunction $\eta, \varepsilon : S \dashv T : \mathscr{A} \to \mathscr{B}$.

PROPOSITION 4.1. *T reflects strong epimorphisms if and only if each* $\varepsilon_A : STA \to A$ *is a strong epimorphism.*

Proof. Since $T\varepsilon_A$ is a retraction by one of the triangular equations for an adjunction, the "only if" part is clear. For the "if" part, we consider $f : A \to B$ where Tf is a strong epimorphism, and form the commutative diagram

$$\begin{array}{ccc} STA & \xrightarrow{\;STf\;} & STB \\ {\scriptstyle \varepsilon_A}\downarrow & & \downarrow{\scriptstyle \varepsilon_B} \\ A & \xrightarrow[\;f\;]{} & B \end{array} .$$

By Section 2 above, STf like Tf is a strong epimorphism since S has a right adjoint; whence $f\varepsilon_A = \varepsilon_B \cdot STf$ is a strong epimorphism, so that f is a strong epimorphism by the results of [5] or of [4].

REMARK 4.2. Clearly we have the same result, for the same reasons, if we replace "strong epimorphism" by "familially-strong epimorphism" or by "epimorphism".

PROPOSITION 4.3. *Consider the following assertions:*

(i) *Each* ε_A *is a strong epimorphism.*

(ii) *T is conservative.*

(iii) *For each A, the map* $\varepsilon_A : STA \to A$ *factorizes through no proper subobject of A.*

Then (i) *implies* (ii) *and* (ii) *implies* (iii), *while* (iii) *implies* (i) *if* $\mathscr{A}$ *admits* (*SEpi, Mon*) *factorizations, or if* $\mathscr{A}$ *admits pullbacks and equalizers, or if* $\mathscr{A}$ *admits pullbacks and pushouts.*

Proof. (i) implies (ii): because ε_A is epimorphic, T is faithful, whence T reflects monomorphisms; but it also reflects strong epimorphisms by Proposition 4.1, and thus reflects isomorphisms. (ii) implies (iii): if $\varepsilon_A = jp$ with j monomorphic, Tj is monomorphic since T has a left adjoint, while Tj is a retraction since $T\varepsilon_A$ is a retraction; thus Tj is invertible, whence j is invertible by hypothesis (ii). Finally (iii) implies (i) in the given conditions by Propositions 3.1 and 3.3.

REMARK 4.4. When each ε_A is a strong epimorphism and *a fortiori* an epimorphism, T is faithful. Without some condition on $\mathscr{A}$, we do not see why a conservative T with a left adjoint need be faithful.

We can now combine Proposition 4.3 with the various results of Section 2 to give conditions under which each ε_A is a familially-strong epimorphism, and then analyze these further using Theorem 3.4 on the existence of $(SEpi, Mon)$ factorizations. The most convenient criteria which emerge are the following:

THEOREM 4.5. *If the conservative* $T : \mathscr{A} \to \mathscr{B}$ *has a left adjoint* S *with counit* $\varepsilon : ST \to 1$, *each* ε_A *is a familially-strong epimorphism if* $\mathscr{A}$ *satisfies any one of the following conditions* (a), (b), (c):

(a) $\mathscr{A}$ *admits pullbacks and pushouts.*

(b) $\mathscr{A}$ *admits non-empty finite limits and arbitrary intersections of monomorphisms.*

(c) $\mathscr{A}$ *admits coequalizers and arbitrary cointersections of strong epimorphisms.*

Moreover, each ε_A *is a small-familially-strong epimorphism if*

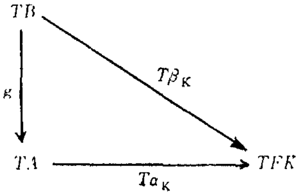
(d) $\mathscr{A}$ *admits small limits.*

Proof. In view of Proposition 4.3, the sufficiency of (a) follows from Proposition 2.4, that of (b) or (c) from Theorem 3.4, and that of (d) from Proposition 2.1.

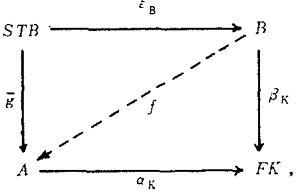
Recall that a functor $T : \mathscr{A} \to \mathscr{B}$ is said to *reflect the limit of* $F : \mathscr{K} \to \mathscr{A}$ if, whenever $(\alpha_K : A \to FK)$ is a cone over F in $\mathscr{A}$ such that $(T\alpha_K : TA \to TFK)$ is a limit-cone in $\mathscr{B}$ for TF, then $\alpha = (\alpha_K)$ is already a limit-cone for F. As we pointed out in the Introduction, it is trivial that a conservative T with a left adjoint reflects the limit of F if F is already known to *admit* a limit in $\mathscr{A}$. The following result, however, requires no prior knowledge of the existence of limits in $\mathscr{A}$:

THEOREM 4.6. *Let* $\eta, \varepsilon : S \dashv T : \mathscr{A} \to \mathscr{B}$ *where each* $\varepsilon_A : STA \to A$ *is a familially-strong [resp. small-familially-strong] epimorphism. Then* T *reflects all limits [resp. all small limits].*

Proof. Let α be any cone over F as above such that $T\alpha$ is a limit-cone for TF, let $(\beta_K : B \to FK)$ be any cone over F, and let g be the unique map rendering commutative



This diagram transforms under the adjunction to the exterior of



where $\bar{g}$ is the transform of g. Since $T\alpha$ is a limit-cone, the family $(T\alpha_K)$ is jointly monomorphic, whence (α_K) is jointly monomorphic because T is faithful. Thus we have a diagonal f as above, so that β does indeed factorize through α, and uniquely so because (α_K) is jointly monomorphic. Thus α is a limit-cone for f.

5. Adjoint-triangle theorems for conservative functors

We consider an adjunction $\eta, \varepsilon : S \dashv T : \mathscr{A} \to \mathscr{B}$ and a functor $P : \mathscr{C} \to \mathscr{A}$ such that $TP : \mathscr{C} \to \mathscr{B}$ has a left adjoint R. We seek conditions on $\mathscr{A}$ and on $\mathscr{C}$ which ensure that P then has a left adjoint Q, provided that T is conservative.

For the moment we suppose only that T is faithful, or equivalently that each ε_A is epimorphic. Write $\zeta : 1 \to TPR$ for the unit of the adjunction $R \dashv TP$, and let the transform of ζ under the adjunction $S \dashv T$ be $\theta : S \to PR$. Consider the injection

$$\mathscr{A}(A, PC) \xrightarrow[T]{} \mathscr{B}(TA, TPC) \xrightarrow[\cong]{} \mathscr{C}(RTA, C), \qquad (5.1)$$

and write $J(A, C)$ for its image.

LEMMA 5.1. *To say that $g : RTA \to C$ is the image under (5.1) of $f : A \to PC$ is exactly to say that we have commutativity in*

$$\begin{array}{ccc} STA & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle \theta_{TA}}\downarrow & & \downarrow{\scriptstyle f} \\ PRTA & \xrightarrow[Pg]{} & PC \end{array} \quad . \qquad (5.2)$$

Thus g lies in $J(A, C)$ if and only if $Pg \cdot \theta_{TA}$ is of the form $f\varepsilon_A$ for some (necessarily unique) $f = f_g$.

Proof. Taking transforms under the adjunction $S \dashv T$, (5.2) is equivalent to

$$\begin{array}{ccc} TA & & \\ {\scriptstyle \zeta_{TA}}\downarrow & \searrow^{Tf} & \\ TPRTA & \xrightarrow[TPg]{} & TPC, \end{array}$$

which precisely expresses that Tf and g correspond under the adjunction $\mathscr{B}(TA, TPC) \cong \mathscr{C}(RTA, C)$.

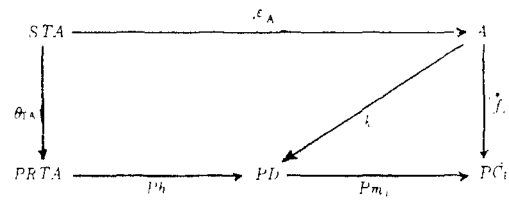
For a given $A \in \mathscr{A}$, write $J(A)$ for the set of all those maps $g : RTA \to C_g$ such that $g \in J(A, C_g)$, and note that, by Lemma 5.1, $J(A)$ is an *ideal*; in the sense that if $g \in J(A)$ and $x : C_g \to C$, we have $xg \in J(A)$. We call this ideal *principal* if there is some *epimorphism* $\gamma_A : RTA \to QA$ in $J(A)$ such that every $g \in J(A)$ is of the form $g = x\gamma_A$ for some $x : QA \to C_g$.

LEMMA 5.2. *P has a left adjoint if and only if, for each A, the* *ideal* $J(A)$ *is principal.*

Proof. To say that P has a left adjoint is to say that, for each A, we have a map $\phi_A : A \to PQA$ such that every $f : A \to PC$ is of the form $Px.\phi_A$ for a unique $x : QA \to C$. Since (5.1) is injective and is natural in C, it comes to the same thing to say that we have some $\gamma_A : RTA \to QA$ in $J(A)$ such that every $g : RTA \to C$ in $J(A)$ is of the form $x\gamma_A$ for a unique $x : QA \to C$. Since $J(A)$ is an ideal, to ask x here to be unique is to ask γ_A to be an epimorphism.

LEMMA 5.3. *Let* $(g_i : RTA \to C_i)_{i \in I}$ *be a family of maps in* $J(A)$, *and let it factorize as* $h : RTA \to D$ *followed by a jointly-monomorphic family* $(m_i : D \to C_i)$. *Then* $h \in J(A)$ *if* ε_A *is a familially-strong epimorphism. If* I *is small* [*resp. if* $I=1$] *it suffices that* ε_A *be small-familially-strong* [*resp. strong*].

Proof. For each g_i there is an f_i such that the corresponding instance of (5.2) commutes, so that we have commutativity of the exterior of



(5.3)

Since TP has a left adjoint, the family (TPm_i) is jointly monomorphic; and since T is faithful because each ε_A is epimorphic, the family (Pm_i) is jointly monomorphic. Thus by the hypothesis on ε_A there is a k rendering commutative the square in (5.3), so that $h \in J(A)$.

Recall from Section 2 above the definition of *weakly cowellpowered*.

THEOREM 5.4. *Let* $\eta, \varepsilon : S \dashv T : \mathscr{A} \to \mathscr{B}$, *let* $P : \mathscr{C} \to \mathscr{A}$, *and let* $R \dashv TP : \mathscr{C} \to \mathscr{B}$. *Then* P *has a left adjoint if any one of the following is satisfied:*

(i) *Each* ε_A *is a familially-strong epimorphism, and* $\mathscr{C}$ *admits coequalizers and arbitrary cointersections of strong epimorphisms.*

(ii) *Each* ε_A *is a small-familially-strong epimorphism, and* $\mathscr{C}$ *is weakly cowellpowered and admits coequalizers and arbitrary cointersections of strong epimorphisms.*

(iii) *Each* ε_A *is a small-familially-strong epimorphism, and* $\mathscr{C}$ *is weakly cowellpowered and admits* $(SEpi, Mon)$ *factorizations and small products.*

Proof. Under the conditions (i) or (ii), $\mathscr{C}$ admits $(SEpi, Mon)$ factorizations by Theorem 3.4; so we may as well assume that $\mathscr{C}$ admits such factorizations. Write $K(A)$ for the subset of $J(A)$ given by those $g: RTA \to C$ in $J(A)$ that are strong epimorphisms, or rather for the set of isomorphism-classes of such strong epimorphisms. If $g \in J(A)$ has the $(SEpi, Mon)$ factorization $g = mh$, it follows from Lemma 5.3 that $h \in J(A)$, so that $h \in K(A)$. It now follows from Lemma 5.2 that P has a left adjoint if there is, for each A, some $\gamma_A: RTA \to QA$ in $K(A)$ through which every $g \in K(A)$ factorizes. Consider the set $(g: RTA \to C_g)$ of all $g \in K(A)$, which is a small set under conditions (ii) or (iii). In each case this has a factorization into a strong epimorphism $\gamma_A: RTA \to QA$ and a jointly-monomorphic family $(m_g: QA \to C_g)_{g \in K(A)}$; by Proposition 3.5 in cases (i) and (ii), and in case (iii) by taking the $(SEpi, Mon)$ factorization of $RTA \to \prod_{g \in K(A)} C_g$. Now Lemma 5.3 gives $\gamma_A \in K(A)$, and the result follows.

REMARK 5.5. The result in case (iii) can be deduced from Theorem 3 of Tholen [7], since when each ε_A is a strong epimorphism the condition (ii) of his theorem merely says that P preserves monomorphisms, while his condition that P preserves small products follows from the fact that TP does so and Theorem 4.6.

REMARK 5.6. When we have only that T is conservative, we get a left adjoint for P by combining the condition on $\mathscr{C}$ given in (i) of Theorem 5.4 with any one of the conditions on $\mathscr{A}$ given in (a), (b), or (c) of Theorem 4.5, or by combining the conditions on $\mathscr{C}$ given in (ii) or (iii) of Theorem 5.4 with any one of the conditions (a), (b), (c), (d) on $\mathscr{A}$ of Theorem 4.5.

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