敎育學會誌(數學敎育)
, Vol. XXIV, No. 3

Journal of the Korea Society of Mathematical Education
Jun. 1986, Vol. XXIV, No. 3

Spectral Theorem and its Application

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ntroduction

n the standpoint of engineering applications, eigenvalue problems among the most .ant problems in connection with matrices, and the number of research papers on ponding numerical methods for computers is enormous. The basic concepts are as s.

$A=(a_{ij})$ be a given square n-rowed matrix and consider vector equation

λX ············(1) where λ is a number.

value of λ for which (1) has a solution $X\neq O$ is called an eigenvalue or character- value of the matrix A.

corresponding solutions $X\neq O$ of (1) are called eigenvectors or characteristic vec- f A corresponding of that eigenvalue λ.

set $\{\lambda_i\}$ is called the spectrum of A.

$|\lambda_i|$ is called the spectral radius of A.

blems of this type occur in connection with physical and technical applications.

Theorem

pose A is a compact self adjoint operator on H(Hilbert space). There exist an or- rmal system φ_1, φ_2, $\cdots$ of eigenvectors of A and corresponding eigenvalues λ_1, λ_2, ch that for all $X\in H$,

$$X=\sum_k \lambda_k <X, \varphi_k> \varphi_k = \sum_{k=1}^{n} \lambda_k <X, \varphi_k> \varphi_k \text{ or } \sum_{k=1}^{\infty} \lambda_k <X, \varphi_k> \varphi_k.$$

$\{\lambda_n\}$ is an infinite sequence, then it converges to zero.

$\{\varphi_n\}$ is an orthonormal basis for H, then the matrix corresponding to A and $\{\varphi_n\}$ e diagonal matrix.

Application

$=(a_{ij})$: Square n-rowed matrix, $AX=\lambda X$.

e set $\{\lambda_k\}$ of the eigenvalues is called the spectrum of A.

e largest of the absolute values of the eigenvalues of A is called the spectral radi- A.

$\frac{\partial^2 u}{\partial t^2}=C^2\frac{\partial^2 u}{\partial x^2}$: the vibrations of an elastic string, where $u(x, t)$ is deflection of the string.

$x=0,\ x=l$: fixed at the ends.

Boundary condition: (2) $u(0,t)=0,\ u(l,t)=0$ for all t.

Initial condition: (3) $u(x,0)=f(x)$, (4) $\left[\frac{\partial u}{\partial t}\right]_{t=0}=g(x)$.

The form $u(x,t)=F(x)G(t)$: solution of (1)

$$\frac{\partial^2 u}{\partial t^2}=FG'',\quad \frac{\partial^2 u}{\partial x^2}=F''G \Rightarrow FG''=C^2F''G \Rightarrow \frac{G''}{C^2G}=\frac{F''}{F}=K.$$

$\therefore$ (a) $F''-KF=0$ (b) $G''-C^2KG=0$

From (2), $u(0,t)=F(0)G(t)$, $u(l,t)=F(l)G(t)=0$ for all t.

(a) $F''-KF=0$: For $K=\mu^2$, $F=Ae^{ux}+Be^{-ux}$, $K=-p^2 \Rightarrow F''+p^2F=0$

$F(x)=A\cos px+B\sin px$: general solution.

$F(0)=A=0$ *and the* $F(l)=B\sin pl=0\ (B\neq 0)$

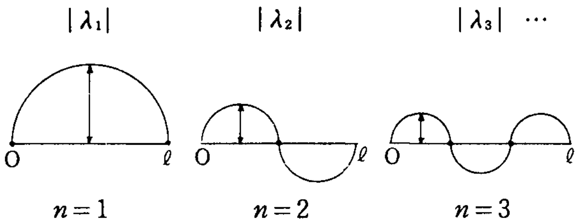
$\therefore \sin pl=0$, $pl=n\pi$ *or* $p=\frac{n\pi}{l}$

Setting $B=1$, $F(x)=F_n(x)$, $F(x)=\sin\frac{n\pi x}{l}$ $(n=1,2,\cdots)$.

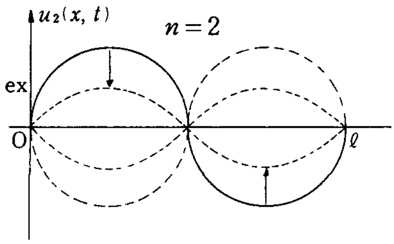
(b) $G''+\lambda_n^2G=0$ $(\lambda_n=cn\pi/\ell)\Rightarrow G_n(t)=B_n\cos\lambda_nt+B_n^*\sin\lambda_nt$: general solution

$\therefore$ $u_n(x,t)=F_n(x)G_n(t)=\sin n\pi x/\ell\,(B_n\cos\lambda_nt+B_n^*\sin\lambda_nt)$ $(n=1,2,\cdots)$.

The set $\{\lambda_1,\lambda_2,\cdots\}$ is called the spectrum.



vibrating string



normal mode for various values

(3) $u(x,0)=\sum_{n=1}^{\infty}B_n\sin n\pi x/\ell=f(x)$ $\left(B_n=\frac{2}{\ell}\int_0^{\ell}f(x)\sin\frac{n\pi}{\ell}x\,dx,\quad n=1,2,\cdots\right)$.

(4) $\left.\frac{\partial u}{\partial t}\right|_{t=0}=\left[\sum_{n=1}^{\infty}(-B_n\lambda_n\sin\lambda_{nt}+B_n^*\lambda_n\cos\lambda_nt)\sin\frac{n\pi x}{\ell}\right]_{t=0}=\sum_{n=1}^{\infty}B_n^*\lambda_n\sin\frac{n\pi x}{\ell}=g$

For $t=0$, $B_n^*\lambda_n=\frac{2}{\ell}\int_0^{\ell}g(x)\sin\frac{n\pi x}{\ell}dx$.

Since $\lambda_n=cn\pi/\ell$, $B_n^*=\frac{2}{cn\pi}\int_0^{\ell}g(x)\sin\frac{n\pi x}{\ell}dx$ $(n=1,2,\cdots)$.

References

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