國數學教育學會誌〈數學教育〉
!86. 6, Vol. XXIV, No. 3

Journal of the Korea Society of Mathematical Education
Jun. 1986, Vol. XXIV, No. 3

Almost Continuous Mappings on an Almost Locally Connected Spaces

by Jong-Moon Shin and Yong-Kil Park
Dongguk University, Gyeongju, Korea

I. Introduction

The object of this paper is to introduce an almost open and almost continuous map-ngs on an almost locally connected spaces. In general, almost continuous mappings re weaker than continuous mappings and stronger than weakly continuous mappings. nd almost open mappings are weaker than open mappings. Noiri [6] and Rose [7] oved the equal condition of weakly continuous mappings. Using it, we studied the con-tion of the mappings under which the image of almost locally connected space is al-ıost locally connected space. The concept of subweakly continuity was introduced by ose. Weakly continuity implies subweakly continuity, but the converse implication does ɔt hold (See Example 2.15). In [6], Noiri proved that the image of weakly continuous apping of connected space is connected. We proved that if Y is second countable, the ıage of subweakly continuous mapping. We found the condition under which the image subweakly continuous mapping of connected space is connected.

II. Preliminaries and Notations

In this paper, (X, τ) will denote a topological space X with a topology τ and all map-ngs are onto.

Definition 2.1. A subset U of a topological space X is said to be regular open (or reg-ar closed) if it is the interior of its own closure (or the closure of its own interior) equivalently, if it is the interior of some closed set (or the closure of some open set). Clearly, a set is regular open if and only if its complement is regular closed.

Definition 2.2. A mapping $f: X \longrightarrow Y$ is said to be almost continuous at a point x in if for every neighborhood V of $f(x)$, there is a neighborhood U of x such that $(U) \subset \mathrm{int}(\mathrm{cl}\, V)$. f is almost continuous on X if f is almost continuous at each point of X.

Theorem 2.3. *f is almost continuous if and only if the inverse image of every reg-lar open subset of Y is open subset of X.* (Theorem 2.2 [5])

f is almost continuous if and only if for each regular open neighborhood V of $f(x)$, here is a neighborhood U of x such that $f(U) \subset V$. (Theorem 2.1 [5])

Definition 2.4. A mapping $f: X \longrightarrow Y$ is said to be weakly continuous if for each ɔint x in X and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $(U) \subset clV$.

Definition 2.5. A mapping $f: X \longrightarrow Y$ is said to be almost open (almost closed) if the image of every regular open (regular closed) subset of X is open (closed) subset of Y.

Clearly, a one-to-one mapping is almost open if and only if it is almost closed.

Definition 2.6. A space (X, τ) is said to be almost regular if for each x in X and each neighborhood U of x, there is a regular open neighborhood V of x such that $\mathrm{cl}\, V \subset \mathrm{int}(\mathrm{cl}\, U)$.

Theorem 2.7. (X, τ): *almost regular if and only if for each x in X and each regular open neighborhood U of x, there is a regular open neighborhood V of x such that $x \in V \subset clV \subset U$.* (Theorem 2.2 [4])

Remark 2.8. Every regular space is almost regular. But the converse is not true. (Example 3.4 [8])

Definition 2.9. A space X is said to be Urysohn space if for every pair of distinct points x and y in X, there are neighborhoods U and V such that $x \in U$, $y \in V$ and $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \phi$

Theorem 2.10. *Every almost regular and Hausdorff space is Urysohn space.* (Theorem 3.2 [4])

Definition 2.11. A space X is said to be almost locally connected at a point x in X if given a regular open neighborhood U of x, there is a connected neighborhood V of x such that $V \subset U$. X is almost locally connected provided X is almost locally connected at each of its points.

Remark 2.12. Every locally connected space is almost locally connected. But the converse is not true. (Example 3.4 [8])

Definition 2.13. A mapping $f: X \longrightarrow Y$ is said to be connected if $f(C)$ is connected whenever C is connected in X.

Definition 2.14. A mapping $f: X \longrightarrow Y$ is said to be subweakly continuous if there is an open basis B for the topology on Y such that $clf^{-1}(V) \subset f^{-1}(clV)$ for all V in B.

Clearly, every weakly continuous mapping is subweakly continuous. But the converse is not true. The following is the example.

Example 2.15. Let X be any set with a non-discrete T_1 topology and let $Y=X$ have the discrete topology. Let $f: X \longrightarrow Y$ be the identity mapping. Then this map is subweakly continuous but not weakly continuous.

Definition 2.16. A space (X, τ) is said to be semi-regular if X has a basis consisting of regular open sets.

III. Almost continuous mappings on an almost locally connected spaces

Lemma 3.1. *A mapping $f: X \longrightarrow Y$ is weakly continuous if and only if $clf^{-1}(V) \subset f^{-1}(clV)$ for each open subset V of Y.*

Theorem 3.2. *If $f: X \longrightarrow Y$ is an almost open and almost continuous mapping and is regular open in Y, then $f^{-1}(V)$ is regular open in X.*

Proof. Let V be a regular open in Y. By Lemma 3.1, $clf^{-1}(V) \subset f^{-1}(clV)$. Since is almost continuous, $f^{-1}(V)$ is open in X. Therefore $\mathrm{int}(\mathrm{cl} f^{-1}(V))$ is regular open in X

nce f is almost open, $f(\mathrm{int}(\mathrm{cl}f^{-1}(V)))$ is open in Y. And
$\mathrm{int}(\mathrm{cl}f^{-1}(V))) \subset f(\mathrm{cl}f^{-1}(V)) \subset ff^{-1}(\mathrm{cl}V) \subset \mathrm{cl}V$
Since $f(int(clf^{-1}(V)))$ is open, $f(int(clf^{-1}(V))) \subset int(clV)=V$.
Hence $int(clf^{-1}(V)) \subset f^{-1}(V)$
Since the converse inclusion is trivial, $f^{-1}(V)$ is regular open.

Theorem 3.3. *Let $f: X \longrightarrow Y$ be an almost open, almost continuous and connected apping. If X is almost locally connected space, then Y is almost locally connected.*

Proof. Let V be a regular open neighborhood of y in Y. Since f is onto, there is x in such that $f(x)=y$. By Theorem 3.2, $f^{-1}(U)$ is regular open neighborhood of x. Since is almost locally connected, there is a regular open connected neighborhood U of x such iat $x \in U \subset f^{-1}(V)$. Since f is almost open and connected, $f(U)$ is open connected neigh- orhood of y in Y and $f(U) \subset ff^{-1}(V) \subset V$.

Corollary 3.4. *If $f: X \longrightarrow Y$ is an almost open, continuous and X is almost locally onnected space, then Y is almost locally connected.*

Theorem 3.5. *If $f: X \longrightarrow Y$ is an almost open, almost continuous, one-to-one mapping nd X is almost regular, then Y is almost regular.*

Proof. Let V be a regular open neighborhood of $f(x)$, By Theorem 3.2, $f^{-1}(V)$ is a egular open neighborhood of x in X. Since X is almost regular, there is a regular open eighborhood U of x in X such that

$$x \in U \subset clU \subset f^{-1}(V) = int(clf^{-1}(V))$$

Since f is almost open and one-to-one, f is almost closed.
Since clU *is regular closed,* $clf(U) \subset f(clU)$.
Hence

$$(x) \in f(U) \subset clf(U) \subset f(clU) \subset f(int(clf^{-1}(V))) \subset int\, f(clf^{-1}(V)) \subset int\, ff^{-1}(clV) \subset int(clV)=V.$$

Corollary 3.6. *Let $f: X \longrightarrow Y$ be an almost open, almost continuous and one-to-one apping. If X is almost regular and Y is Hausdorff space, then Y is Urysohn space.*

Proof. By Theorem 3.5, Y is almost regular. By Theorem 2.10, Y is Urysohn space.

Theorem 3.7. *If $f: X \longrightarrow Y$ is a weakly continuous mapping and Y is almost regular, hen f is almost continuous.*

Proof. Let V be a regular open neighborhood of $f(x)$. Since Y is almost regular, there is a regular open neighborhood W of $f(x)$ such that $f(x) \in W \subset clW \subset V$. Since f is weakly ontinuous, there is a neighborhood U of x such that

$$f(x) \in f(U) \subset clW \subset V$$

But f can not be continuous mappings in Theorem 3.7. The following is the example.

Example 3.8. Let R be the set of real numbers and let $\mathcal{U}$ be the usual topology on R nd let τ be the another topology on R generated by the union of $\mathcal{U}$ and τ, the topology f countable complement on R. Then (R, τ) is an almost regular space. Define $f: (R, \mathcal{U}) \longrightarrow (R, \tau)$ be the identity mapping. Then f is weakly continuous. Hence f is lmost continuous. But f is not continuous at any point.

Corollary 3.9. *Let $f: X \longrightarrow Y$ be a weakly continuous mapping. If Y is almost reg- ılar and semi-regular, then f is continuous.*

Proof. By Theorem 3.7, f is almost continuous. Since Y is semi-regular, f is continuous.

Definition 3.10. A collection $\mathscr{U}$ of subsets of X is locally finite if each x in X has a neighborhood meeting only finite many $U\in\mathscr{U}$

Lemma 3.11. *If* $\{A_\lambda|\lambda\in\Lambda\}$ *is a locally finite system, then* $\bigcup clA_\lambda = cl\bigcup A_\lambda$

Theorem 3.12. *If* $f: X\longrightarrow Y$ *is subweakly continuous,* X *is connected and* $\{f^{-1}(W_i)|W_i\in B,\ B$ *is a base of* $Y\}$ *is a locally finite system and* Y *is second countable then* Y *is connected.*

Proof. Suppose Y is not connected. Then there are non-empty disjoint open subsets V_1 and V_2 in Y such that $V_1\cup V_2=Y$.

Hence $f^{-1}(V_1)\cap f^{-1}(V_2)=\phi$ *and* $f^{-1}(V_1)\cup f^{-1}(V_2)=X$.

Since Y is second countable, there is a basis B such that $V_1=\bigcup_{j=1} W_{1j}$ and $V_2=\bigcup_{j=1} W_{2j}$, $W_{ij}\in B(i=1,2)$

$$cl\ f^{-1}(V_1)=cl\ f^{-1}(\bigcup_{j=1} W_{1j})=cl\bigcup_{j=1} f^{-1}(W_{1j})=\bigcup_{j=1} cl\ f^{-1}(W_{1j})\subset\bigcup_{j=1} f^{-1}(clW_{1j})$$
$$=f^{-1}(\bigcup_{j=1} clW_{1j})\subset f^{-1}(cl\bigcup W_{1j})=f^{-1}(V_1)$$

Therefore $f^{-1}(V_1)=cl\ f^{-1}(V_1)$

Hence $f^{-1}(V_1)$ is open and closed subset of X. Similarly, $f^{-1}(V_2)$ is open and closed subset of X. It contradicts that X is connected.

References

[1] C.W. Baker, *Properties of subweakly continuous functions,* Yokohama Mathematical Journal Vol. 32 (1984), 39~43.

[2] F. Siwiec, *Countable spaces having one non-isolated point,* Proc. Math. Soc. 57(1976 345~348.

[3] J. Dugundji, *Topology,* Allyn and Bacon, Inc., Boston, (1966).

[4] M.K. Singal and S.P. Arya, *On almost regular spaces,* Glasnik Math. 4 (1969), 89~99.

[5] M.K. Singal and A.R. Singal, *Almost continuous mappings,* Yokohama Math. J 16 (1968), 63~73.

[6] Takashi Noiri, *On weakly continuous mappings,* Proc. Amer. Math. Soc. 46 (1974) 120~124.

[7] D.A. Rose, *Weak continuity and almost continuity,* International Journal of Mathematics and Mathematical Sciences.

[8] Vincent J. Mancuso, *Almost locally connected spaces,* J. of Austral. Math. Soc. Series A 31 (1981), 421~428.

[9] P.E. Long and D.A. Carahan, *Comparing almost continuous functions,* Proc Amer. Math. Soc. 38 (1973), 413~418.

[10] Stephen Willard, *General Topology,* Addison-Wesley Publishing Company.