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Primary Ideals and Valuation ideals.

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I. Introduction

Let D be an integral domain, let $\mathcal{2}$ denote the set of primary ideals, and let $\mathcal{V}$ denote the set of valuation ideals of D. The object of this paper is to investigate the significance of the relationships $\mathcal{V} \subseteqq \mathcal{2}$, $\mathcal{2} \subseteqq \mathcal{V}$, and Our point of departure was the observation in that f D is a Dedekind domain, then $\mathcal{2} = \mathcal{V}$.

II. Preliminary results on valuation ideals.

Definition 2. 1. An ideal A of a domain D is called a valuation ideal if there exists a valuation ring $D_v \supseteqq D$ and an ideal A_v of D_v such that $A_v \cap D = A$

Definition 2. 2. If A is ideal of D and if S is the set of all nontrival valuations of he quotient field K of D which are non-negative on D, then $A' = \cap_{v \in s} A \cdot D_v$ is called he complition of A. If $A = A'$, then A is called complete. $D' = \cap_{v \in s} D_v$ is the integral closure of D (8, p. 15, Theorem 6)

Proposition 2. 3. *Let D be a domain. If A is any ideal of D, denote by A′ the ompletion of A and by A* the intersection of those valuation ideals of D which contain A. Then*

(1) $A' \cap D = A^*$

(2) $(x)' = xD'$ *for any* $x \in D$.

(3) *If* $(x) = (x)'$ *for some* $x \in D$, $x \neq 0$, *then* $D = D'$.

(4) $(x) = (x)^*$ *for all* $x \in D$ *if and only if* $D = D'$.

Definition 2. 4. An integral domain D is a Prüfer domain if each nonzero finitely generated ideal of R is invertible.

Theorem 2. 5. *If every ideal of D is an intersection of primary ideals and if every primary ideal is a valuation ideal, then D is Prüfer.*

III. Main Theorem.

Theorem 3. 1. $\mathcal{V} \subseteqq \mathcal{2}$ *if and only if every proper prime ideal of D is maximal.*

Proof. Suppose every prime ideal of D is maximal, and let A be a valuation. Then there exists a valuation ring $D_v \supseteqq D$ and an ideal A_v of D such that $A_v \cap D = A$. If P is the center of D_v on D, then $D \subseteqq D_P \subset D_v$ and D is a one dimensional quasi-local ring. Therefore $A_v \cap D_P = A'$ is primary; and since $A' \cap D = A$, A is also primary. Conversely, assume $\subseteqq$ 2, and suppose there exist prime ideals P, P' of D such that $0 \subset P \subset P' \subset D$. By (6, p. 37), there exists a valuation ring D_v having prime ideals P_v, P'_v which lie over P, P' respectively. Choose $x \in P'$, $x \notin P$ and $y \neq 0$ in P, and let $A = (XY)\ D_v \cap D$. Then A is a valuation ideal and $A \subseteqq P$. Claim; A is not primary. For if A is primary, $xy \in A$ and $x \notin P$ implies $y \in A$. But then $y = rxy$ for some $r \in D_v$, and hence $1 = rx \in P'_v$, a contradiction.

Lemma 3. 2. *Let M be a prime ideal of a domain D, and suppose there exists a prime ideal $P \subset M$ such that there is no prime ideal P_1 whth $P \subset P_1 \subset M$.*
then P is the intersection of the M-primary ideals of D which contain P.

Theorem 3. 3. *Let D be a quasi-local domain, and suppose for any nonzero prime ideal P of D there exists a prime ideal $N(P) \subset P$ such that if P_1 is prime ideal $\subset P$, then $P_1 \subseteqq N(P)$. Then D satisfies the a. c. c. for prime ideals and the prime ideals of D are linearly ordered.*

Proof. If $P_1 \subset P_2 \subseteqq \cdots$ is anascending chain of prime ideals of D, then $U = \cup P_i$ is also prime, so if $U \neq P_i$ for all i, thne $P_i \subseteqq N(U)$ for all i; and we would have $U = \cup P_i \subseteqq N(U)$ $\subset U$, a contradiction. Threrefore, D satisfies the a. c. c for prime ldeals. Now suppose there exist prime ldeals P_1, P_2 of D such that $P_1 \nsubseteqq P_2$ and $P_2 \nsubseteqq P_1$. Since D has the a. c. c. for prime ldeals, there exists a prime ideal M, maximal with respect to the properties $P_1 \subseteqq M$, $P_2 \nsubseteqq M$, Since $P_2 \nsubseteqq M$, M is not the maximal ideal of D and there exist a prime ide $M_a \supset M$. If $\{M_a\}$ is the set of all such prime ideals, then $M \neq \cap M_a$ since $P_2 \subseteqq M_a$ and P_2! M. Therefore, by Zorns lemma, there is a prime ideal M_0 minimal with respect to the property that $M_0 \supset M$. Therefore, $M \subseteqq N(M_0) \subset M_0$ implies $M = N(M_0)$. But then $P_2 \subset M_0$ means $P_2 \subseteqq N(M_0) = M$, a contradiction to the choice of M.

Corollary 3. 4. *Let D be a quasi-local domain such that D satisfies the a. c. c. for prime ideals If $2 \subseteqq v$, then the prime ideals of D are linearly ordered.*

Lemma 3. 5. *Let D be a quasi-local domain which satisfies the for prime ideals, an suppose $2 \subseteqq v$. Then D is integrally closed.*

Theorem 3. 6. *Let D be a domain which satisfies the a. c. c. for prime ideals If $2 \subseteqq v$, then D is a Prüfer domain.*

Proof. It is sufficient to see D_P is a valuation ring for any prime ideal P of D. Therefore we may assume that D is quasi-local, and D is integrally closed. Suppose the there exists nonzero $x, y \in D$ such that x/y and $y/x \notin D$. x, y are then nonunits of D, s the fact that the prime ideals of D are ordered implies $V(x, y)$ is prime. Consider the the set of all prime ideals of D which are of the form for such x, y. By the a. c. c, coi tains a maximal element P and suppose x, y are the elements of the above type such th $P = V(x, y)$. $(x^2, y^2) \cdot D_P$ is then primary and hence a valuation ideal. Therefore $xy \in (x^2$ $y^2) \cdot D_P$. We may assume that $x/y \in D_P$. Then $x/y = r/s$, $r, s \in D$, $s \notin P$. But this mea

r/s, $s/r \notin D$, and $s \notin P$ implies $V(r, s) \supset P$, a contradiction to the choice of P.

Corollary 3. 7. *A noetherian domain D has the property* $2 \subseteqq V$ *if and only if D is Dedekind domain.*

Proof. D is Dedekind domain if and only if D is a noetherian Prüfer domain. now apply 3. 6.

Corollary 3. 8. *Let D be a noetherian domain and let P be a prime ideal of D such that every P-primary ideal is valuation ideal. Then P is minimal prime of D and* P_p *is rank 1, discrete valuation ring.*

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