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Uniqueness of the Cauchy Problem for Certain Quasi-linear Partial Differential Equations

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I. Introduction

Since Holmgren's uniqueness theorem in 1901 the uniqueness theorem of the Cauchy pro-em for partial differential operators has been variously developed and improved. The first ›neral results of the noncharacteristic Cauchy problem were given in 1957 by A. P. Cal-›ron [1] as an application of singular integral operators, what is called "pseudodifferen-ıl operators". Thereafter, C. Zuily has improved them to more general results. In this ·ticle we prove the uniqueness theorem for operators which are given by operator in C. uily's context adding some nonlinear terms.

II. Main Results

In a neighborhood V of a point x_0 *in* R^n let

$$S=\{x\epsilon V:\ \phi(x)=\phi(x_0)\}$$

› a C^∞-hypersurface and $P=\sum_{|\alpha|\leq m} a_\alpha(x)\ D^\alpha$

differential operator of order m whose principal symbol $P_m(x,\xi)$ has C^∞ coefficients in . Here we usually assume that S will be noncharacteristic for P at x_0, which means that, ith $N_0=d\phi(x_0)$,

$$P_m(x_0\ N_0)\neq 0.$$

Now we need some conditions on P_m.

(i) There exist a conic neighborhood ${}^\Gamma N_0$ and m functions $\lambda_j(x,\xi,N)$ which are C^∞ in $V\times R^n\ O\times{}^\Gamma N_0$ such that for every ξ nonparallel to N,

$$P_m(x,\xi+\tau N)=P_m(x,N)\ \prod_{j=1}^{m}(\tau-\lambda_j\ (x,\xi,N))$$

in $V\times R^n\backslash 0\times{}^\Gamma N_0$.

(ii) If a root $\lambda_j(x,\xi,N)$ is real (complex) at a point, it remains real (complex) at every point.

(iii) Real roots are simple and the complex ones are at most double.

ow we represent the basic inequality given by L. Nirenberg [3] and C. Zuily [6].

Theorem 1. *Suppose that the operator P satisfies the conditions (i) - (iii). Then* *there exist positive constants C, T_0, k_0, r such that for $T \leqq T_0, k \geqq k_0$ we have*

$$\sum_{|\alpha| \leqq m-1} \int_0^T e^{k(t-T)^2} \|D^\alpha v\|^2 dt \leqq C\left(\frac{1}{k}+T^2\right) \int_0^T e^{k(t-T)^2} \|Pv\|^2 dt$$

for every $v \in C^\infty$ with supp $v \subset \{(x, t): 0 \leqq t \leqq T, |x| \leqq r\}$.
Here $\|.\|$ stands for the $L^2(R^n_X)$ - norm.

Consider a nonlinear partial differential equations of order 2 as following.

$$P(x, D)u = P_2(x, D)u + \sum_{|\alpha| \leqq 1} a_\alpha(x) D^\alpha u + f(u) = g(x) \quad \cdots\cdots\cdots (1)$$

Here, f is a continuous function satisfying

$$|f(u_1) - f(u_2)| \leqq C \sum_{|\alpha| \leqq 1} |D^\alpha u_1 - D^\alpha u_2|$$

for some positive constant C and a_α and g are bounded and C^∞ *in V.*

Proposition 2. *Suppose that the principal part P_2 of P is real, elliptic, and C^∞ in V. Then the following is true:*

If u_1 and u_2 are two solutions of (1) *such that*

$$\left(\frac{\partial}{\partial n}\right)^j u_1 = \left(\frac{\partial}{\partial n}\right)^j u_2 \quad \text{on } S \text{ for } j=0,\ 1$$

where n is outward nomal to S.
Then u_1 and u_2 coincide in some neighborhood W of x_0.

Proof. (a) First we reduce this problem to simple one.
Since

$$P(x, D)u_1 = P(x, D)u_2$$

$$P_2(x, D)\ (u_1 - u_2) + \sum_{|\alpha| \leqq 1} a_\alpha(x)\ D^\alpha(u_1 - u_2) + f(u_1) - f(u_2) = 0.$$

Then we get

$$|P_2(x, D)(u_1 - u_2)| \leqq C \sum_{|\alpha| \leqq 1} |D^\alpha(u_1 - u_2)|$$

$$\left(\frac{\partial}{\partial n}\right)^j (u_1 - u_2)|_S = 0 \quad \text{for } j=0,\ 1.$$

Therefore we obtain the following reduced one;

$$|P_2(x, D)u| \leqq C \sum_{|\alpha| \leqq 1} |D^\alpha u| \quad \cdots\cdots\cdots (2)$$

$$\left(\frac{\partial}{\partial n}\right)^j u|_S = 0 \quad \text{for } j=0,\ 1.$$

Then u vanishes near x_0.

Now assuming $x_0 = 0$, we can, by the classical Holmgren's transformation and the noncharacteristicity of S, get

$$S = \{(x, t) \mid t = C|x|^2\}.$$

On the other hand, if u denotes the new solution under studying, we have

$$supp\ u \subset \{(x, t);\ t \geqq C|x|^2\}.$$

(b) Secondly we verify that the principal part P_2 satisfies the condition (i) - (iii). Assume that

$$P_2\ (x, \xi + \tau N) = P(x, N)\,[\,\tau^2 + a(x, \xi)\,\tau + b(x, \xi)\,]$$

for every ξ nonparallel to N.
Since P_2 is real and C^∞

$$\lambda_1 = \frac{-a(x, \xi) + \sqrt{a(x, \xi)^2 - 4b(x, \xi)}}{2}$$

$$\lambda_2 = \frac{-a(x, \xi) - \sqrt{a(x, \xi)^2 - 4b(x, \xi)}}{2}$$

are two C^∞ roots.
Since $a(x, \xi)$ and $b(x, \xi)$ are real, if λ_1 and λ_2 are real (complex) at a point, then they must remain real (complex) at every point.
On the other hand, the complex roots, if any, are of multiplicity 1 but no real root occurs since P_2 is elliptic. Therefore all conditions are satisfied.

(c) Now we prove the uniqueness result.
Since our solution $u(x, t)$ does not have compact support in $\{(x, t);\ 0 < t < T,\ |x| \leqq r\}$, we can not directly apply Theorem 1.

Let $\theta(t)$ be a C^∞ real function such that

$$\theta(t) = \begin{cases} 0 & t > T \\ 1 & t \leqq T_1 < T \end{cases}$$

nd $0 \leqq \theta(t) \leqq 1$.
'ake $0 < T_2 < T_1$. Putting $\nu(x, t) = \theta(t)\,u(x, t)$,

$$supp\ \nu \subset \{(x, t)\,|\,0 < t < T,\ |x| \leqq r\}.$$

'hen applying Theorem 1 to $\nu = \theta u$, we get

$$\sum_{|\alpha| \leqq 1} \int_0^T e^{k(t-T)^2}\ \|D^\alpha(\theta u)\|^2\,dt$$

$$\leqq C\Big(\frac{1}{k} + T^2\Big) \int_0^T e^{k(t-T)^2}\ \|P_2\ (\theta u)\|^2\,dt$$

$$\leqq C\Big(\frac{1}{k} + T^2\Big) \int_0^T e^{k(t-T)^2}\ \|\theta P_2\,u\|^2$$

$$+ C\Big(\frac{1}{k} + T^2\Big) \int_0^T e^{k(t-T)^2}\ \|[P_2, \theta]u\|^2\,dt.$$

rom (2) we have

$$\int_0^T e^{k(t-T)^2}\ \|\theta P_2\,u\|^2 \leqq C \sum_{|\alpha| \leqq 1} \int_0^T e^{k(t-T)^2}\ \|D^\alpha u\|^2\,dt.$$

onsequently we get

$$\sum_{|\alpha| \leqq 1} \int_0^{T_2} e^{k(t-T)^2}\ \|D^\alpha u\|^2\,dt \leqq C\Big(\frac{1}{k} + T^2\Big) \sum_{|\alpha| \leqq 1} \int_{T_2}^{T_1} e^{k(t-T)^2}\ \|D^\alpha u\|^2\,dt$$

$$+C\left(\frac{1}{k}+T^2\right)\int_{T_1}^{T_2} e^{k(t-T)^2}\,\|[P_2,\theta]u\|^2\,dt.$$

Estimating the weight function $e^{k(t-T)^2}$ we get

$$\sum_{|\alpha|\leq 1}\int_0^{T_2}\|D^\alpha u\|^2\,dt \leq C\left(\frac{1}{k}+T^2\right)\sum_{|\alpha|\leq 1}\int_{T_2}^{T}\|D^\alpha u\|$$

$$+C\left(\frac{1}{k}+T^2\right)e^{k((T_1-T)^2-(T_2-T)^2}\int_{T_1}^{T}\|\,\|[D_2,\theta]u\|^2\,dt.$$

Letting k go to ∞ and T to 0 we deduce that

$$u=0 \quad \textit{for} \quad 0\leq t\leq T_2,$$

Here we use the notation C as a positive constant which may not be the same one in different occurences.

Proposition 3. *Let $Q(x,\xi)$ be the elliptic symbol of homogeneous degree 2 with real and C^∞ coefficients. Consider the following equations:*

$$P(x,D)u=Q^2(x,D)u+\sum_{|\alpha|\leq 3} a_\alpha D^\alpha u+f(u)=g(x),$$

Where a_α, g and f are in the same class as in proposition 2.
Then the given equation has the uniqueness property as in proposition 2.

Proof. Let u_1 and u_2 be two solutions which coincide on the noncharacteristic hyper surface S. Put $P_4=Q^2$.
Then we get

$$|P_4(x,D)\ (u_1-u_2)\ |\leq C\sum_{|\alpha|<3}|D^\alpha(u_1-u_2)|.$$

Thus this problem can be reduced to the one as in proposition 2. On the other hand,

$$P_4\ (x,\xi+\tau N)=P_4\ (x,N)\ \ [\tau^2+a(x,\xi)\tau+b(x,\xi)]^2$$

for every ξ nonparallel to N and for some $a(x,\xi)$, $b(x,\xi)$ real, C^∞ functions.
But since P_4 is elliptic, it has no real root and complex ones are at most double. Other conditions in Theorem 1 are trivially satisfied. Therefore similar method as in proposition 2 gives our result.

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