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On The Radom-Nikodym Theorem

by Kim Sung-Zai

Kyung Hee University, Seoul, Korea

I. Introduction

In this paper, we define the measure ν will be finite if and only if f is integrable using the notion absolute continuity in sign measure, and investigate some properties of measurable space. We also obtain the following results. An atempt will be made to establish and verify concretely the more general framework in which the discussion of the Radom-Nikodym theorem by absoluty continuity still make sense in abstract measurable space.

II. Preliminary

Definition 1. Let μ be a signed measure on $(X, \mathcal{M})$ and let (A, B) be a Hahn decomposition of X for μ. Define μ^+, μ^- and $|\mu|$ on by : $\mu^+(E) = \mu(E \cap B)$, $\mu^-(E) = -\mu(E \cap B)$ and $|\mu(E)| = \mu^+(E) + \mu^-(E) < +\infty$ for all $E \in \mathcal{M}$. The set functions μ^+, μ^- and $|\mu|$ are called the positive variation of μ, the nagative variation of μ, and the total variation of μ, respectively.

Definition 2. An arbitrary measure $\nu: \mathcal{M} \to R^*$ is said to be absolutely continuous with respect to μ, in symbols, $\nu \ll \mu$, if and only if, for every $E \in \mathcal{M}$, $\mu(E) = 0$ implies $\nu(E) = 0$. If ν is defined by a nonnegative measurable function $f: X \to R^*$ and μ, then it follow from $\nu(E) = 0$ that $\nu \ll \mu$. In cass μ is $\sigma-$finite, then the converge is also true.

Proposition 1. *If μ is signed measure on the measurable set $(X, \mathcal{M})$,* then there exist two disjoint sets A and B whose union is X, such that $\mu^+(B) = 0$, $\mu^-(A) = 0$.

Proposition 2. *Let μ be a signed measure on the measurable space $(X, \mathcal{M})$. Then $\mu(E) = \mu^+(E) - \mu^-(E)$ for every measurable set $E \in \mathcal{M}$ and μ is also bounded.*

Proposition 3. *Let μ and ν be signed measures on $(X, \mathcal{M})$ and $\mu(E)$ be the Jordan decomposition of ν, then the following are equivalent:*

(a) $\nu \ll \mu$ (b) $\nu^+ \ll \mu$, $\nu^- \ll \mu$

Propersition 4. *If μ and ν are total finite measures such that $\nu \ll \mu$ and, is not identically zero, then there exist a positive number ε and a measurable set E such that $\mu(E) > 0$ and such that E is a positive set for the signed measure $\nu - \varepsilon\mu$.*

III. Main Theorem

Theorem (Radon-Nikodym Theorem) *Let* $(X, \mathcal{M}, \mu)$ *be a* σ*-finite measure space, and let* ν *be a measure defined on* $\mathcal{M}$ *which is absolutely continuous with respect to* μ*. Then there is nonnegative measurable function* f *such that for each set* E *in* $\mathcal{M}$ *we have*

$$\nu(E) = \int_E f(s)\mu(ds).$$

The function f *is unique in the sense that if* g *is any measurable function with this property then* $g=f$ *a. e.* $[\mu]$.

Proof. Let P be the set of nonnegative integral functions K such that

$$\int_E h(s)\mu(ds) \qquad E \in \mathcal{M}.$$

The set P may be partially ordered by defining $h \leqq g$ to mean that $h(s) \leqq g(s)$ almost everywhere. We will show, using Zorn's lemma, that P contains a maximal element. Indeed, let Q be a totally ordered subfamily of P and

$$\alpha = \sup_{h \in Q} \int_S h(s)\mu(ds), \text{ then } 0 \leqq \alpha \leqq \mu(S).$$

Let h_n be a sequence of element of Q such that

$$\int_S h_n(s)\mu(ds) \leqq \int_S h_{n+1}(s)\mu(ds) \to \alpha.$$

Then, since Q is totally ordered, it follows that $h_n(s) \leqq h_{n+1}(s)$ almost everywhere and thus without loss of generality, we may assume that $h_n(s) \leqq h_{n+1}(s)$ everywhere.
$h(s) = \lim_n h_n(s)$ is integrable,

$$\int_S h(s)\mu(ds) = \alpha, \text{ and } h_n \leqq h, \quad n = 1, 2 \cdots\cdots\cdots.$$

To see that h is an upper bound for Q, let g be an arbitrary element of Q. Then either $g \leqq h_n$ for some n, in which case

$g \leqq h$ or $g \geqq h_n$ for every n, in which case $g \geqq h$ and

$$0 \geqq \int_S g(s)\mu(ds) - \alpha = \int_S [g(s) - h(s)]\mu(ds) \geqq 0,$$

and this implies that $g(s) = h(s)$ almost everywhere. Thus h is an upper bound for Q and Zorn's lemma shows the existence of a maximal element f in P.

Let $\nu_1(E) = \nu(E) - \int_E f(s)\mu(ds), \qquad E \in \mathcal{M}.$

Then ν_1 is a μ-continuous finite nonnegative measure on $\mathcal{M}$.

To complete the proof we demonstrate that $\nu_1(E) = 0$ for every E in $\mathcal{M}$. If this is not so, then $\nu_1(s) > 0$ and there is a positive number k such that

(1) $\mu(s) - k\,\nu_1(s) < 0.$

By the Hahn decomposition there is a set A in $\mathcal{M}$, such that

$$\mu(EA) - k\nu_1(EA) \leqq 0, \quad \mu(EA^c) - k\nu_1(EA^c) \geqq 0, \quad E \in \mathcal{M},$$

and then, a fortiori,

(2) $\mu(EA) - k\nu_1(EA) \leqq 0, \quad \mu(EA^c) - k\nu_1(EA^c) \geqq 0, \quad E \in \mathcal{M},$

therefore

(3) $\dfrac{1}{k}\mu(EA) - \nu_1(E) \leqq 0, \quad E \in \mathcal{M}.$

If $\mu(A) = 0$ then $\nu_1(A) = 0$ and $\mu(S) = \mu(A^c)$,
$\nu_1(S) = \nu_1(A^c)$: thus from (1) and (2) we have

$$0 \leqq \mu(A^c) - k\nu_1(A^c) = \mu(S) - k\nu_1(S) < 0$$

a contradiction. This proves that $\mu(A) > 0$.

Let g be defined by the equations:

$$g(s) = 1/k \quad s \in A, \quad g(s) = 0 \quad s \in A.$$

The inequality (3) may then be written

$$\int_E g(s)\mu(ds) \leqq \nu_1(E) = \nu(E) - \int_E f(s)\mu(ds) \quad E \in \mathfrak{m}.$$

Which show that

$$\int_E [f(s) + g(s)]\mu(ds) \leqq \nu(E) \qquad E \in \mathfrak{m}.$$

Since $f + g > f$, this contradicts the maximality of f in P.
Therefore $\nu_1(E) = 0$ for every E in $\mathfrak{m}$, On the other hand, if f and g are two μ-integrable functions such that

$$\nu(E) = \int_E f(s)\mu(ds) = \int_E g(s)\mu(ds),$$

then f and g differ by a μ- null function. This establishes the uniqueness of f and completes the proof of the Theorem.

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