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Stability of Closed set in Flow

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In [1], the following Theorem shows the connection between stability and Lyapunov func-
n.

Theorem 1. *A closed set M is stable if and only if there exists a function $\varphi(x)$ fined on X with the following properties:*
$\varphi(x)$ if and only if $x \in M$,
For every $\varepsilon > 0$, there is a $\delta > 0$ such that $\varphi(x) \geqq \delta$ whenever $\rho(x, M) \geqq \varepsilon$; also for y sequence (x_n), $\varphi(x_n) \to 0$ whenever $x_n \to x \in M$,
$\varphi(xt) \leqq \varphi(x)$ for all $x \in X$, $t \geqq 0$.

Here, in general, a function $\varphi(x)$ need not be continuous. Such an example is given in]. However, in this paper, even though a function $\varphi(x)$ is continuous, we will prove it a closed set M is stable if and only if there exists a continuous function $\varphi: X \to R^+$ ch that (1) for any $x \in X$, $\varphi(x) = 0$ if and only if $x \in M$, (2) for any $x \in X$ and $t \in R^+$, $xt) \leqq \varphi(x)$.

Here, X and R^+ mean an arbitrary metric space and the set of non-negative reals, spectively. A closed set M of X is stable if for each $\varepsilon > 0$ and $x \in M$, there is a $\delta = \delta$ $\varepsilon) > 0$ such that $S(x, \delta) R^+ \subset S(M, \varepsilon)$. A point $x \in X$ is ***positively Lyapunov stable*** if each $\varepsilon > 0$, there is a $\delta > 0$ such that $\rho(x, y) < \delta$ implies $\rho(xt, yt) < \varepsilon$ for $t \in R^+$. If ery point of X is positively Lyapunov stable, X is called positively Lyapunov stable.

Theorem 2. *Let X be positively Lyapunov stable. Then a subset M of X is closed l positively invariant if and only if there exists a continuous function $\varphi: X \to R^+$:h that (1) for each $x \in X$, $\varphi(x) = 0$ if and only if $x \in M$*
(2) for each $x \in X$ and $t \in R^+$, $\varphi(xt) \leqq \varphi(x)$.

Proof. (1) Sufficiency. Let $x \in \bar{M}$. Then there exists a sequence (x_n) in M such that $\to x$. Since a function φ is continuous, $\varphi(x_n) \to \varphi(x)$. By the assumption, $\varphi(x_n) = 0$. early, $\varphi(x) = 0$. Also, $x \in M$. Hence M is closed.

To see that M is positively invariant, let $x \in M$ and $t \in R^+$. Then $\varphi(x) = 0$. Since $0 \leqq \varphi$ $) \leqq \varphi(x)$, $\varphi(xt) = 0$. Thus $xt \in M$. As $t \in R^+$ was arbitrary, M is positively invariant.

2) Necessity. Let M be positively invariant. We define the function $\varphi: X \to R^+$ by ting $\varphi(x) = \sup_{t \in R^+} \dfrac{\rho(xt, M)}{1 + \rho(xt, M)}$. Then $\varphi(x)$ is defined on X.

Let $x \in M$. For any $t \in R^+$, $xt \in M$. Thus $\rho(xt, M) = 0$. Consequently, $\dfrac{\rho(xt, M)}{1+\rho(xt, M)} = 0$.

This shows that $\varphi(x) = \sup\limits_{t \in R^+} \dfrac{\rho(xt, M)}{1+\rho(xt, M)} = 0$. Assume that $\varphi(x) = 0$, for any $x \in X$.

Let $x \notin M$. Then $\rho(x, M) > 0$. Here, $\varphi(x) = \sup\limits_{t \in R^+} \dfrac{\rho(xt, M)}{1+\rho(xt, M)} \geqq \dfrac{\rho(x, M)}{1+\rho(x, M)} > 0$.

This contradicts to the fact that $\varphi(x) = 0$. Therefore, $x \in M$.

We claim that $\varphi(xt) \leqq \varphi(x)$, for any $x \in X$ and $t \in R^+$. Now,

$$\varphi(xt) = \sup_{s \in R^+} \frac{\rho(xt(s), M)}{1+\rho(xt(s), M)} = \sup_{s \in R^+} \frac{\rho(x(t+s), M)}{1+\rho(x(t+s), M)} = \sup_{s \in (t, +\infty)} \frac{\rho(xs, M)}{1+\rho(xs, M)}$$

$$\leqq \sup_{s \in R^+} \frac{\rho(xs, M)}{1+\rho(xs, M)} = \varphi(x).$$ Hence $\varphi(xt) \leqq \varphi(x)$.

In order to prove the continuity of $\varphi: X \to R^+$, let $x \in X$. Sinee X is positively Lyapunov stable, for any $\varepsilon > 0$, there exists $a\delta > 0$ such that $\rho(x, y) < \delta$ implies $\rho(xt, yt) < \varepsilon$, for all $t \in R^+$. For any $z \in M$, $\rho(xt, M) \leqq \rho(xt, z) \leqq \rho(xt, yt) + \rho(yt, z)$.
Then $\rho(xt, M) - \rho(xy, yt) \leqq \rho(yt, z)$. Also, $\rho(xt, M) - \rho(xt, yt) \leqq \rho(yt, M)$. Thus $\rho(xt, M) \leqq \rho(yt, M) + \rho(xt, yt) < \rho(yt, M) + \varepsilon$. Here, $\varphi(x) = \dfrac{\rho(xt, M)}{1+\rho(xt, M)} < \dfrac{\rho(yt, M)+\varepsilon}{1+\rho(yt, M)+\varepsilon}$

$$< \frac{\rho(yt, M)+\varepsilon}{1+\rho(yt, M)} < \frac{\rho(yt, M)}{1+\rho(yt, M)} + \varepsilon \leqq \varphi(y) + \varepsilon.$$

Therefore, $\varphi(x) \leqq \varphi(y) + \varepsilon$. Similarly, $\varphi(y) \leqq \varphi(x) + \varepsilon$. This implies that $|\varphi(y) - \varphi(x)| \leqq$
Hence the function $\varphi: x \to R^+$ is continuous.

Theorem 3. *Let X be positively Lyapunov stable and let M be a closed subset of X. Then M is positively invariant if and only if M is stable.*

Proof. Let M be positively invariant. Let $x \in M$. Then $xt \in M$. In view of positively Lyapunov stabllity of X, for any $\varepsilon > 0$, there exists $a\ \delta > 0$ such that $\rho(x, y) < \delta$ implies $\rho(xt, yt) < \varepsilon$, for any $x \in M$ and $t \in R^+$. Now, $\rho(yt, M) \leqq \varphi(yt, xt) < \varepsilon$. Clearly, $yt \in S(M,$ $S(M, \varepsilon)$. Thus $S(x, \delta) R^+ \subset S(M, \varepsilon)$. This means that M is stable.

Conversely, let M be stable. Let $x \in M$ and $t \in R^+$. If $xt \notin M$, $\rho(xt, M) = \varepsilon > 0$. Sinc M is stable, $xt \in S(M, \varepsilon)$. Then $\rho(xt, M) < \varepsilon$. This is a contradiction. Hence $xt \in M$. Consequently, M is positively invariant. This completes the proof.

Remark. *From Theorem2 and Theorem 3, a closed set M of X is stable if and onl if there exists a continuous function $\varphi: X \to R^+$ such that* (1) *for any $x \in X$, $\varphi(x) =$ if and only if $x \in M$,* (2) *for any $x \in X$ and $t \in R^+$, $\varphi(xt) \leqq \varphi(x)$.*

References

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[2] K. S. Sibirsky, *Introduction to Topological Dynamics,* Noordorff, Leyden, Netherlands, 1975.