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Necessary and Sufficient Conditions on Pointwise Conformal Deformation of Scalar Curvatures of Some metrics

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I. Introduction

In this paper, we consider the problem of describing the set of scalar curvature functions associated with Riemannian metrics on a given connected, but not necessarily orientable, compact manifold of dimension$\geqq 3$.

We shall call metrics g and g_1 pointwise conformal if $g_1=p(x)g$ for some positive function $p\in C^\infty(M)$. Now if a given metric g on M, where dim $M=n\geqq 3$, has scalar curvature k and we seek K as the scalar curvature of the metric $g_1=u^{4/(n-2)}g$ pointwise conformal to g, then $u>0$ must satisfy (1.1) $\frac{4(n-1)}{n-2}\triangle u-ku+Ku^{(n+2)/(n-2)}=0$,

where $\triangle$ is the Laplacian in the g metric.

In carrying out our analysis of (1.1), the sign of the lowest eigenvalue $\lambda_1(g)$ of the linear part of (1.1), in other words,

(1.2) $$L\phi=-\frac{4(n-1)}{n-2}\triangle\phi+k\phi=\lambda_1(g)\phi,$$

plays a prominant part because the sign of $\lambda_1(g)$ is conformal invariant.

In this paper our results are proved in the case of $\lambda_1(g)<0$. And for basic existence theorems, we use the method of upper and lower solutions [1, pp370-371].

II. Main results

Let M be a compact connected n-dimensional manifold, which is not necessarily orientable and possesses a given Riemannian structure g. We denote the volume element of this metric by dV, the gradient by ∇, and function f on M is written $\bar{f}$, that is,

$$\bar{f}=\frac{1}{vol(M)}\int_M fdv.$$

We let $H_{s,p}(M)$ denote the Sobolev space of functions on M whose derivatives through order s are in L_p. The norm on $H_{s,p}(M)$ will be denoted by $\|\ \|_{s,p}$. The usual $L_2(M)$ inner product will be written $<,\ >$.

Lemma 1. *Assume $K<0$. Then K is the scalar curvature of some metric pointwise conformal to the given metric g if and only if $\lambda_1(g)<0$.*

Proof. See Theorem 4.1 in [4].

The above Lemma 1 shows that if $\lambda_1(g)<0$, then one can always pointwise conformally deform g to a metric of constant negative scalar curvature. Hence we can without loss of generality restrict our attention to the case where the given metric already has a constant negative scalar curvature $k=-c$, where $c>0$ is a constant. Thus (1.1) reads

(2.1) $$\frac{4(n-1)}{n-2}\triangle u+cu=-Ku^{(n+2)/(n-1)},\quad u>0.$$

In order to understand (2.1), one must first free it from geometric considerations and consider the equation

(2.2) $$-Lu=\triangle u+ru=-Hu^a,\quad u>0,$$

where $a=$ constant>1, $r=$ constant>0 and $H\in C^\infty(M)$. Throughout this paper, we will assume that all data (M, metric g, and curvature K) are smooth merely for convenience.

J. L. KAZDAN and F. W. WARNER showed that if $\lambda_1(g)<0$ and $\bar{H}<0$, then there is a constant $0<r_0(H)\leqq\infty$ such that one can solve (2.2) for $0<r<r_0(H)$, but not for $r>r_0(H)$. And they also showed that if $r_0(H)=\infty$, then $H(x)\leqq 0$ for all $x\in M$. In fact, they proved that if $H(x_0)>0$ for some $x_0\in M$, then $r_0(H)<\infty$ [4]. Now we will prove that if $H(x)\leqq 0$ for all $x\in M(H\not\equiv 0)$, then $r_0(H)=\infty$.

Lemma 2 (Existence of lower solutions). *Let $h, H\in Lp$ with $p>\dim M$. If $\lambda_1<0$, then given any positive continuous function u on M, there is a function $u_-\in H_{2,p}$ with $0<u_-\leqq u$ satisfying $Lu_-\leqq Hu_-^a$, that is, $\triangle u_- -hu_- +Hu_-^a\geqq 0$.*

Proof. See Lemma 2.8 in [4].

Theorem (Existence of upper solutions). *If $H\leqq 0(H\not\equiv 0)$, then (2.2) has an upper solution for any positive constant r, so $r_0(H)=\infty$.*

Proof. If we show that $Lu_+\geqq Hu_+^a$ for some positive function $u_+>0$ and any positive cosstant r, that is, $\triangle u_+ +ru_+ +Hu_+^a\leqq 0$, then Lemma 2 implies the existence of solution of $\triangle u+ru+Hu^a=0$, so $r_0(H)=\infty$. Let r be a positive constant. If we put $u_+=e^\psi$, then $\triangle u_+=e^\psi(\triangle\psi+|\nabla\psi|^2)$. Hence

$$\triangle u_+ +ru_+ +Hu_+^a\leqq 0$$

if and only if $\triangle\psi+|\nabla\psi|^2+r+He^{c\psi}\leqq 0$

for some function ψ and $c>0$.

If $Lv=-\triangle v-aHv$, then

$$\lambda_1=\min_{v\not\equiv 0}\frac{\langle Lv,v\rangle}{\|v\|_2^2}=\min_{v\not\equiv 0}\frac{\|\nabla v\|_2^2+\langle v,-aHv\rangle}{\|v\|_2^2}\quad[4,\ \text{p. }117].$$

If $H\leqq 0(\not\equiv 0)$, then $-H\geqq 0(\not\equiv 0)$.

Hence there is a constant $a>0$ such that $\lambda_1=r$.

Therefore $\triangle\Psi+aH\Psi=-r\Psi(\Psi>0)$ [4, p. 117].

Put $\Psi=e^u$. Then

$$\triangle u+|\nabla u|^2+r+aH=0.$$

Define $\psi=u+\lambda$. Then

$$\begin{aligned}&\triangle\psi+|\nabla\psi|^2+r+He^{c\psi}\\&=\triangle u+|\nabla u|^2+r+He^{cu+c\lambda}\\&=-aH+He^{cu+c\lambda}\\&=H(e^{cu+c\lambda}-a)\leqq 0\end{aligned}$$

for sufficiently large λ, since $H\leqq 0$.

This proves our Theorem.

Remark. In [2] or [3], *the similar cases for Gaussian curvatures on compact 2-dimensional manifolds are treated.*

References

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