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On The Final Convergence Structure

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I. Introduction

In this paper we introduce a notion of final convergence structure and investigate some properties. For notions, not given here, the reader is asked to refer to [1], [2], [3], [5] and [8]. For a set X, $F(X)$ denotes the set of all filters on X and $P(X)$ the set of all subsets of X. For each $x \in X$, x is the principal ultrafilter containing $\{x\}$.

A convergence structure on X is a map q from $F(X)$ into $P(X)$ satisfying the following conditions:

(1) for each $x \in X$, $x \in q(x)$;

(2) for $F, G \in F(X)$, if $F \subset G$, then $q(F) \subset q(G)$;

(3) if $x \in q(F)$, then $x \in q(F \cap x)$.

The pair (X, q) is called a convergence space. If $x \in q(F)$, we say that F q-converges to x. The filter $V_q(x)$ obtained by intersecting all filters which q-converges to x is called the q-neighborhood filter at x. If $V_q(x)$ q-converges to x for each $x \in X$, then q is called a pretopology, and (X, q) a pretopological space. Pretopology q is called a topology if for each $x \in X$, the filter $V_q(x)$ has a filterbase $B_q(x) \subset V_q(x)$ with the following condition:

$y \in G(x) \in B_q(x)$ implies $G(x) \in B_q(y)$.

II. Preliminaries

For a convergence sturcture q to be a limitierung, it is necessary and sufficient that the following condition be satisfied:

$x \in q(F)$ and $x \in q(G)$ implies $x \in q(F \cap G)$.

For q to be a pseudo-topology, the following additional condition is necessary and sufficient:

if F' q-converges to s for all ultrafilters F' finer than F, then F q-converges to x.

A convergence structure q is said to be a weakli uniformizable if there exists a set Q of completely regular topologies such that $q = inf_c Q$.

For any convergence space (X, q). let $(\rho X, \rho(q))$ be the convergence space defined on the same underlying sets as follows:

F $\rho(q)$-converges to x if and only if G q-converges to x for each ultrafilter G finer than F.

Then the space $(\rho X, \rho(q))$ is the finest pseudo-topological space coarser than X, and it is called the pseudo-topological modification of X. Note that X and ρX have the same ultrafilter convergence. A convergence space (X, q) is said to be an almost pseudo-topological if $q(F)=\rho(q)(F)$ for all ultrafilter F on X, i. e. X and ρX have the same ultrafilter convergence. Convergence space (X, q) and (Y, p) are said to be a pseudo-topologically coherent if $\rho(X\times Y)=\rho X\times\rho Y$.

Let f be a map from a convergence space (X, q) onto a convergence space (Y, p). If $F\in F(X)$, then $f(F)$ will denote the filter on Y generated by $\{f(F)\,|F\in F\}$. f is said to be continuous at a point $x\in X$ if for any filter F q-converges to x, the filter $f(F)$ p-converges to $f(x)$. If f is continuous at every point $x\in X$, then f is said to be continuous. The mapping f is called a convergence quotient map if p is the finest convergence structure on Y relative to which f is continuous.

Theorem 2. 1. ([1]) *Let (X, q) be a convergence space. $\pi(q)$ denoted by: F $\pi(q)$-converges to x if and only if $F\subset Vq(x)$ for each $x\in X$. Then $\pi(q)$ is the finest pretopology coarser than q.*

Theorem 2. 2. ([4]) *The following statements about f are equvalent:*

(1) f is a convergence quotient map;

(2) A filter F p-converges to y in Y if and only if there is x in $f^{-1}(y)$ and $G\in F(X)$ such that $F\geqq f(G)$ and G q-converges to x.

Theorem 2. 3. ([2]) *A convergence structure q is weakly uniformizable if and only if F q-converges to x whenever $x\in \cap F$ and $q(F)\neq\phi$*

Let (X, q) be a convergence space. We may associate with q the set function I_q defined for a given A X by

$$I_q(A)=\{x\in A\,|A\in V_q(x)\}.$$

Then the set $\{I_q(A)\,|A\subset X\}$ is a base for the topology $\psi(q)$ on X.

Theorem 2. 4. ([2]) *Let q be a weakly uniformizable convergence structure. If q is a limitierung, then $\pi(q)$ and $\psi(q)$ are weakly uniformizable.*

Theorem 2. 5. ([7]) *Convergence space (X, q), (Y, p) are almost pseudo-topological if and only if the pair X, Y is pseudo-topological coherent.*

III. Final convergence structure

Let X be a set, (X_α, q_α) be a convergence space for each $\alpha\in\Lambda$. f be a map from a convergence space (X_α, q_α) onto X. q is a map from $F(X)$ into $P(X)$ satisfying the following condition:

for each element x of X, $F\in F(X)$, F q-converges to x if and only if for each $\alpha\in\Lambda$, $f_\alpha^{-1}(F)$ q_α-converges to x_α for some $x_\alpha\in f_\alpha^{-1}(x)$. Then we obtain a convergence structure q on X that is said to be final convergence structure induced by the family $\{f_\alpha|\alpha\in\Lambda\}$.

Throughout this section, q mean final convergence structure on X induced by the family $\{f_\alpha|\alpha\in\Lambda\}$, where f_α is a map from convergence space (X_α, q_α) onto a set X for each $\alpha\in$

The following facts can be easily verified.

Proposition 3. 1. *q is the finest of all convergence structure on X which allow very f_a to be continuous for each $a \in A$.*

Corollary 3. 2. *If s is final convergence structure induced by the family $\{f_a | a \in A\}$, ıen every f_a is convergence quotient map.*

Proposition 3. 3. *A map f from the convergence space (X, q) onto a convergence space $Y, p)$ is continuous if and only if $f \circ f_a$ is continuous for each $a \in A$.*

Lemma 3. 4. *For each $a \in A$, $x_a \in f_a^{-1}(x)$*

$$f_a^{-1}(Vq(x)) \subset Vq_a(x_a).$$

Proof.
$$\begin{aligned} f_a^{-1}(Vq(x)) &= f_a^{-1}(\cap \{F \in F(x) | x \in q(F)\} \\ &= \cap \{f_a^{-1}(F) | F \in F(X),\ x \in q(F)\} \\ &= \cap \{f_a^{-1}(F) | F \in F(X),\ f_a^{-1}(x) \subset q_a(f_a^{-1}(F))\} \\ &\subset \cap \{G \in F(X_a) | x_a \in q_a(G)\} \\ &= Vq_a(x_a). \end{aligned}$$

Proposition 3. 5. *Every convergence space of the family $\{(X_a, q_a) | a \in A\}$ is pretoɔlogical, then q is a pretopology.*

Proof. Since $Vq_a(x_a) \subset f_a^{-1}(Vq(x))$ for some $x_a \in f_a^{-1}(x)$ and q_a is a pretopology for ach $a \in A$,

$$x_a \in q_a(Vq_a(x_a)) \subset q_a(f_a^{-1}(Vq(x))).$$

hrrefore $Vq(x)$ q-converges to x. i. e. q is a pretopology.

Proposition 3. 6. *If (X_a, q_a) is a limitierung space, f_a injective for each $a \in A$, then is a limitierung.*

Proof. If F_i q-converges to x, $i=1, 2$, then $f_a^{-1}(F_i)$ q_a-converges to $f_a^{-1}(x)$ *for each* $\in A$, $i=1, 2$. Since q_a is a limitierung for each $a \in A$ and

$$f_a^{-1}(F_1) \cap f_a^{-1}(F_2) = f_a^{-1}(F_1 \cap F_2),$$

$_a^{-1}(F_1 \cap F_2)$ q_a-converges to $f_a^{-1}(x)$. Thus $F_1 \cap F_2$ q-converges to x. Hence q is limitierung.

Proposition 3. 7. *Let $f: (S, r) \to (T, p)$ be a injective convergence quotient map. f r is limitierung, then p is limitierung.*

Proof. Let $F_1, F_2 \in F(T)$. If F_i p-converges to t $i=1, 2$. By theorem 2. 2, there are $G_i \in F(S)$ such that $f(G_i) \leq F_i$ and G_i r-converges to $F^{-1}(t)$, $i=1, 2$. Since r is limitierung, $G_1 \cap G_2$ r-converges to $f^{-1}(t)$. Since

$$f(G_1 \cap G_2) \subset f(G_1) \cap f(G_2) \subset F_1 \cap F_2,$$

$F_1 \cap F_2$ p-converges to t. Hence p is a limitierung.

Proposition 3. 8. *If (X_a, q_a) is a weakly uniformizable convergence space for each $\in A$, then q is a weakly uniformizable.*

Proof. Let $x \in \cap F$ and $q(F) \neq \phi$ then F q-converges to some $x' \in X$. ence $f_a^{-1}(F)$ q_a-converges to $x'_a \in f_a^{-1}(x')$ for each $a \in A$. Since $x \in \cap F$,

$$f_a^{-1}(x) \subset f_a^{-1}(\cap F) = \cap f_a^{-1}(F).$$

ince q_a is a weakly uniformizable, $f_a^{-1}(F)$ q_a-converges to $x_a \in f_a^{-1}(x)$.

Thus F q-converges to x. By theorem 2.3, q is a weakly uniformizable.

Corollary 3.9. *If q_α is a limitierung and weakly uniformizable for each $\alpha\in\Lambda$, then $\pi(q)$, $\psi(q)$ are weakly uniformizable.*

hereafter, (X_α, q_α) means a compact convergence space such that for each, $F\in F(X_\alpha$ $\alpha_{q_\alpha}(F)$ is one-point set for each $\alpha\in\Lambda$.

Proposition 3.10. *If (X_α, q_α) is a pseudo-topological space for each $\alpha\in\Lambda$, then q is a pseudo-topology.*

Proof. Let (X_α, q_α) be a pseudo-topological space for each $\alpha\in\Lambda$. Given a filter F on X, let F' q-converges to x for all ultrafilter F', finer than F, then $f_\alpha^{-1}(F')$ q_α-converges to $x_\alpha\in f_\alpha^{-1}(x)$ for each $\alpha\in\Lambda$. Since $f_\alpha^{-1}(F)\subset f_\alpha^{-1}(F')$, X_α is a compact and $\alpha_{q_\alpha}(f_\alpha^{-1}$ $F))$ $=\{x_\alpha\}$, $f_\alpha^{-1}(F)$ q_α-converges to x_α for each $\alpha\in\Lambda$. By definition of final convergence structure, F q-converges to x. Therefore (X, q) is a pseudo-topological space.

Lemma 3.11. *Let (X_α, q_α) be a convergence space, f_α be a map from (X_α, q_α) onto a set X and g_α be a map from $(X_\alpha, \rho(q_\alpha))$ onto X defined by $f_\alpha=g_\alpha$ in underlying sets for each $\alpha\in\Lambda$. If q^* is final convergence structure on X induced by the family $\{g_\alpha|\alpha\in\Lambda\}$, then*

$$q^*\leqq\rho(q)\leqq q.$$

Proof. Let F be a filter on X, if F q-converges to x, then

$$x_\alpha\in g_\alpha^{-1}(x)=f_\alpha^{-1}(x),$$

$$x_\alpha\in q_\alpha(f_\alpha^{-1}(F))\subset\rho(q_\alpha)\ (f_\alpha^{-1}(F))=\rho(q_\alpha)\ (g_\alpha^{-1}(F)).$$

Since q^* is final convergence structure induced by the family $\{g_\alpha|\alpha\in\Lambda\}$, F q^*-converges to x. Thus $q^*\leqq q$. Since $\rho(q)$ is the finest pseudo-topology coarser than q and by proposition 3.10, q^* is a pseudo-topology, $q^*\leqq\rho(q)\leqq q$.

Proposition 3.12. *If (X_α, q_α) is an almost pseudo-topological space for each $\alpha\in\Lambda$, then q is an almost pseudo-topology.*

Proof. Let q^* be the final convergence structure defined in lemma 3.11, If F q^*-converges to x, then $g_\alpha^{-1}(F)$ $\rho(q_\alpha)$-converges to $x_\alpha\in g_\alpha^{-1}(x)$, Since (X_α, q_α) is an almost pseudo topological space, for all ultrafilter F on X,

$$x_\alpha\in g_\alpha^{-1}(x)=f_\alpha^{-1}(x),$$

$$x_\alpha\in\rho(q_\alpha)\ (g_\alpha^{-1}(F))=q_\alpha(f_\alpha^{-1}(F)).$$

Thus, F q-converges to x. Hence

$$q^*(F)\subset q(F).$$

By lemma 3.11, for all ultrafilter F on X,

$$q^*(F)=\rho(q)\ (F)=q(F).$$

Thus, (X, q) is an almost pseudo-topological space.

From theorem 2.5 and proposition 3.12, we can obtain the following corollary.

Corollary 3.13. *If pair (X_1, X_2) of convergence spaces is a pseudo-topologically coherent, then (Y, p) is an almost pseudo-topological space, where p is final convergence structure induced by the family $\{f_i|f_i\colon (X_i, q_i)\longrightarrow Y;$ surjection, $i=1, 2\}$.*

Lemma 3. 14. *Let f_a be a map defined in lemma 3, 11, and g_a be a map from $(X_a, (q_a))$ onto X defined by $g_a = f_a$ in underlying sets for each $a \in \Lambda$. If q' is final convergence structure on X induced by the family $\{g_a | a \in \Lambda\}$, then*

$q' \leqq \pi(q) \leqq q$.

Proof. For each $F \in F(X)$, if F q-converges to x, then

$x_a \in g_a^{-1}(x) = f_a^{-1}(x)$,

$x_a \in q_a(f_a^{-1}(F)) \subset \pi(q_a)\ (f_a^{-1}(F)) = \pi(q_a)(g_a^{-1}(F))$.

ince q' is final convergence structure induced by the family $\{g_a | a \in \Lambda\}$, F q'-converges to . Thus $q' \leqq q$. By proposition 3. 5, q' is a pretopology. Since $\pi(q)$ is the finest pretopogy coarser than q,

$q' \leqq \pi(q) \leqq q$.

Proposition 3. 15. *If (X_a, q_a) is an almost pretopological space for each $a \in \Lambda$, then X, q) is an almost pretopological space.*

Proof. Let F be any ultrafilter on X. If F q'-converges to x, then

$x_a \in g_a^{-1}(x) = f_a^{-1}(x)$,

$x_a \in \pi(q_a)(g_a^{-1}(F)) = \pi(q_a)(f_a^{-1}(F))$

r each $a \in \Lambda$, where q' is the final convergence structure on X defined by lemma 3. 14. ince (X_a, q_a) is an almost pretopological space for each $a \in \Lambda$, $f_a^{-1}(F) = g_a^{-1}(F)$ q_a-conerges to x_a. Thus F q-converges to x, i. e. $q'(F) \subset q(F)$ for all ultrafilter F on X. By mma 3. 14, $q(F) \subset \pi(q)(F) \subset q'(F)$ for any filter F on X. Therefore for all ultrafilter on X,

$q'(F) = \pi(q)(F) = q(F)$.

hus q is an almost pretopology.

References

.] D. C. Kent, Convergence functions and their related topologies, *Fund. Math. LIV*, 125-133, (1964).

:] ______ , On convergence greups and convergence uniformities, *Fund. Math. LX*, 213-222, (1967).

] ______ , A note on pretopologies, *Fund. Math. LXII*, 95-100, (1968).

] ______ , Convergence quotient maps, *Fund. Math. LXV*, 197-205, (1969).

] ______ and G. D. Richardson, Minimal convergence spaces, *Trans. A. M. S.* 160, 487 -499, 1971.

] ______ , Locally compact convergence spaces, *Michigan Math. J.* 22, 353-360, (1975).

] ______________ , Some product theorems for convergence spaces, *Math. Nachr.* 87, 43-51, (1979).

] J. W. Nam and H. I. Choi, Pseudo-topological coherence for convergence spaces, *Pusan Kyungnam Math. J.* 2, 54-60, (1986).

] S. Willard, *General topology*, Addison-Weselly Pub. Co. Inc., (1970).