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Some Properties of Pseudo-Topological Spaces

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I. Introduction

In 1964, D. C. Kent introduced convergence functions and their related topologies, and D. C. Kent and G. D. Richardson investigated some properties of convergence spaces and applied the concepts as product of convergence spaces in 1975. In discussing product of convergence spaces, they restricted to finite product.

The purpose of this paper is that most of the results extend to arbitrary product using the initial convergence structure.

We start section II with some preliminary results related to this paper.

In scction III, we study properties of almost pseudo-topological spaces, and define the pseudo-topological coherence and investigate its properties in section IV.

II. Preliminaries

Our notation and terminology will coincide with that of [1], [2], and [3].

For a set X, $F(X)$ denotes the set of all filters on X and $P(X)$ the set of all subsets of X. For each $x \in X$, x is the principal ultrafilter containing $\{x\}$.

A convergence structure on X is a map q from $F(X)$ into $P(X)$ satisfying the following conditions:

(1) for each $x \in X$, $x \in q(x)$;

(2) for F, $G \in F(X)$, if $F \subset G$, then $q(F) \subset q(G)$;

(3) if $x \in q(F)$, then $x \in q(F \cap x)$.

The pair (X, q) is called a convergence space. If $x \in q(F)$, we say that F q-converges to x.

A function f of convergence space (X, q) onto convergence space (Y, p) is said to be continuous at x if for every filter F q-converges to x, $f(F)$ p-converges to $f(x)$.

Let X be a set, (X_α, q_α) be a convergence space for each $\alpha \in \Lambda$, f_α be a function X onto (X_α, q_α). The initial convergence space q on X induced by the family $\{f_\alpha \mid \alpha \in \Lambda\}$ is a map from $F(X)$ into $P(X)$ satisfying the following condition:

for any element $x \in X$, $F \in F(X)$, $x \in q(F)$ if and only if $f_\alpha(F)$ q_α-converges to $f_\alpha(x)$ for each $\alpha \in \Lambda$.

III. Almost pseudo-topological spaces

In this section, we shall investigate some properties of arbitrary pseudo-topological space. Throughout this paper, for each $\alpha\in\Lambda$, $(X\alpha, q\alpha)$ means compact convergence space such that $\mathfrak{a}_{q_\alpha}(F)$ is one-point set for each $F\in F(X\alpha)$, (X, q) means initial convergence space induced by the family $\{f_\alpha|\alpha\in\Lambda\}$, where $f\alpha$ is a map X onto convergence space $(X\alpha, q\alpha)$ for each $\alpha\in\Lambda$, and $(\Pi X\alpha, q')$ means initial convergence space induced by the family $\{P\alpha|\alpha\in\Lambda\}$, where $P\alpha$ is canonical projection of $\Pi X\alpha$ onto $X\alpha$ for each $\alpha\in\Lambda$.

A pseudo-topological space (S, r) is a convergence space with the property that F r-converges to s whenever each ultrafilter finer than F r-converges to s.

Proposition 3.1. *If (X_α, q_α) is a pseudo-topological space for each $\alpha\in\Lambda$, then (X, q) is a pseudo-topological space.*

Proof. Let $(X\alpha, q\alpha)$ be a pseudo-topological space for each $\alpha\in\Lambda$. Given a filter F on X, let F' q-converge to x for all ultrafilter F' finer than F. Then $f_\alpha(F')$ q_α-converges to $f_\alpha(x)$ for each $\alpha\in\Lambda$. Since $f_\alpha(F)\subset f_\alpha(F')$, $X\alpha$ is a compact and $\mathfrak{a}_{q_\alpha}(f_\alpha(F))=\{f_\alpha(x)\}$, $f\alpha(F)$ $q\alpha$-converges to $f\alpha(x)$ for each $\alpha\in\Lambda$. By definition of initial convergence structure, F q-converges to x. Thus (X, q) is a pseudo-topological space.

Corollary 3.2. *If $(X\alpha, q\alpha)$ is a pseudo-topological space for each $\alpha\in\Lambda$, then $(\Pi X\alpha, q')$ is a pseudo-topological space.*

For any convergence space (S, r), let $(\rho S, \rho(r))$ be the convergence space defined on the same underlying sets as follows:

F $\rho(r)$-converges to s if and only if G r-converges to s for each ultrafilter G finer than F.

Then the space $(\rho S, \rho(r))$ is the finest pseudo-topological space coarser then S, and it is called the pseudo-topological modification of S. Note that S and ρS have the same ultrafilter convergence.

A convergence space (S, r) is said to be an almost pseudo-topological if $r(F)=\rho(r)(F)$ for all ultrafilter F on S, i.e. S and ρS have the same ultrafilter convergence.

Proposition 3.3. *$\Pi\rho X\alpha\leqq\rho\Pi X\alpha\leqq\Pi X\alpha$ where $(X\alpha, q\alpha)$ is a convergence space for each $\alpha\in\Lambda$.*

Proof. By definition of pseudo-topological modification, $\rho\Pi X_\alpha\leqq\Pi X_\alpha$. We shall show that $\Pi\rho X_\alpha\leqq\rho\Pi X_\alpha$. If F $\rho(q')$-converges to $(x_\alpha)_{\alpha\in\Lambda}$ in $\rho\Pi X_\alpha$, then F' q'-converges to $(x_\alpha)_\alpha$ in ΠX_α for all ultrafilter F' finer than F. That is, $P_\alpha(F')$ q_α-converges to x_α in X_α for each $\alpha\in\Lambda$. Since $\rho X_\alpha\leqq X\alpha$, $P_\alpha(F')$ $\rho(q_\alpha)$-converges to x_α in ρX_α for each $\alpha\in\Lambda$. Thus, F' converges to $(x_\alpha)_{\alpha\in\Lambda}$ in $\Pi\rho X_\alpha$. By corollary 3.2, $\Pi\rho X_\alpha$ is a pseudo-topological space. Hence F converges to $(x_\alpha)_{\alpha\in\Lambda}$ in $\Pi\rho X_\alpha$. Therefore $\Pi X_\alpha\leqq\rho\Pi X_\alpha\leqq\Pi X_\alpha$.

Lemma 3.4. *Let (X_α, q_α) be convergence space for each $\alpha\in\Lambda$, let f_α be a map from X onto (X_α, q_α), g_α be a map from X onto $(X_\alpha, \rho(q_\alpha))$ defined by $f_\alpha=g_\alpha$ in underlying sets. If q^** is initial convergence structure on *X induced by the family $\{g_\alpha|\alpha\in\Lambda\}$, then $q^*\leqq\rho(q)\leqq q$, i.e. $q(F)\subset\rho(q)(F)\subset q^*(F)$ for each $F\in F(X)$.*

Proof For each $F\in F(X)$, if $x\in q(F)$, then
$g_\alpha(x)=f_\alpha(x)\in q_\alpha(f_\alpha(F))\subset\rho(q_\alpha)\ (f_\alpha(F))=\rho(q_\alpha)\ (g_\alpha(F))$.
nce q^* is initial convergence structure induced by $\{g_\alpha|\alpha\in\Lambda\}$, $x\in q^*(F)$. Thus $q^*\leqq q$. By oposition 3. 1, q^* is a pseudo-topological because $\rho(q_\alpha)$ is a pseudo-topological. Since ρ is the finest pseudo-topological coarser than q, hence $q^*\leqq\rho(q)\leqq q$.

Proposition 3. 5. *If (X_α, q_α) is an almost pseudo-topological for each $\alpha\in\Lambda$, then $(X,$ is an almost pseudo-topological.*

Proof. Let q^* be the initial convergence structure defined in lemma 3. 4, if $x\in q^*(F)$, n for each $\alpha\in\Lambda$, $g_\alpha(x)\in\rho(q_\alpha)\ (g_\alpha(F))$. Since (X_α, q_α) is an almost pseudo-topologi-, for all ultrafilter F on X, $f_\alpha(x)=g_\alpha(x)\in\rho(q_\alpha)\ (f_\alpha(F))=q_\alpha(f_\alpha(F))$. Thus $x\in q(F)$, nce $q^*(F)\subset q(F)\subset\rho(q)\ (F)$ for all ultrafilter F on X. By lemma 3. 4, $q(F)=q^*(F)=\rho$ (F) for all ultrafilter F on X. Thus, (X, q) is an almost pseudo-topological.

Corollary 3. 6. *If (X_α, q_α) is an almost pseudo-topological for each $\alpha\in\Lambda$, then $X_\alpha, q')$ is an almost pseudo-topological.*

IV. Pseudo-topological coherence

n this section, we shall investigate relations and properties of pseudo-topological cohe-nce and almost pseudo-topological spaces.

onvergence spaces (S, r) and (T, t) are said to be a pseudo-topologically coherent if $S\times T)=\rho S\times\rho T$.

Definition 4. 1. $(X_\alpha)_{\alpha\in\Lambda}$ of convergence space X_α is said to be a pseudo-topologically erent if $\rho\Pi X_\alpha=\Pi\rho X_\alpha$.

Lemma 4. 2. *Let X_α be a convergence space. Then X_α is an almost pseudo-topologi- if and only if $(X_\alpha)_{\alpha\in\Lambda}$ is a pseudo-topologically coherent.*

Proof. Suppose that $\Pi\rho X_\alpha<\rho\Pi X_\alpha$, there exists a filter F on ΠX_α such that F con-ges to $(x_\alpha)_{\alpha\in\Lambda}$ in $\Pi\rho X_\alpha$ and does not in $\rho\Pi X_\alpha$. By definition of $\rho\Pi X_\alpha$, there exists ultrafilter F' such that F' does not q'-converge to $(x_\alpha)_{\alpha\in\Lambda}$ in ΠX_α and $F'\geqq F$. But F' verges to $(x_\alpha)_{\alpha\in\Lambda}$ in $\Pi\rho X_\alpha$ because F converges to $(X_\alpha)_{\alpha\in\Lambda}$ in $\Pi\rho X_\alpha$ and $F'\geqq F$. us $P_\alpha(F')$ $\rho(q_\alpha)$-converges to x_α in ρX_α for each $\alpha\in\Lambda$, so that $P_\alpha(F')$ q_α-converges x_α in X_α for each $\alpha\in\Lambda$ by definition of almost pseudo-topological, hence F' q'-converges $(x_\alpha)_{\alpha\in\Lambda}$ in ΠX_α. This contradicts the fact that an ultrafilter F' does not q'-converge $(x_\alpha)_{\alpha\in\Lambda}$ in ΠX_α, thus $(X_\alpha)_{\alpha\in\Lambda}$ is a pseudo-topologically coherent. Conversely, suppose t $(X_\alpha)_{\alpha\in\Lambda}$ is a pseudo-topologically coherent, and let F_α be arbitrary ultrafilter on X_α. F_α q_α-converges to x_α in X_α for each $\alpha\in\Lambda$, then F_α $\rho(q_\alpha)$-converges to x_α in ρX_α each $\alpha\in\Lambda$. On the other hand, if F_α $\rho(q_\alpha)$-converges to x_α in ρX_α for each $\alpha\in\Lambda$, n ΠF_α converges to $(x_\alpha)_{\alpha\in\Lambda}$ in $\Pi\rho X_\alpha$. Since $\rho\Pi X_\alpha=\Pi\rho X_\alpha$, ΠF_α $\rho(p')$-converges to $)_{\alpha\in\Lambda}$ in $\rho\Pi X_\alpha$ and so F q'-converges to $(x_\alpha)_{\alpha\in\Lambda}$ in ΠX_α for all ultrafilter F finer than α, thus $P_\alpha(F)$ q_α-converges to x_α in X_α for each $\alpha\in\Lambda$. Since $P_\alpha(F)$ and F_α are afilters on X_α and $P_\alpha(F)\geqq P_\alpha(\Pi F_\alpha)=F_\alpha$, $P_\alpha(F)=F_\alpha$ q_α-converges to x_α in X_α for each $\alpha \in \Lambda$. Thus X_α and ρX_α have the same ultrafilter convergence, hence for each $\alpha \in$ X_α is an almost pseudo-topological.

Proposition 4. 3. *If $(X_\alpha)_{\alpha \in \Lambda}$ is a pseudo-topologically coherent, then (X, q) is an almost pseudo-topological space.*

Proof. From lemma 4. 2 and proposition 3. 5, the proof of proposition 4. 3 is satisfied.

Corollary 4. 4. *If $(X_\alpha)_{\alpha \in \Lambda}$ is a pseudo-topologically coherent, then ΠX_α is an almo: pseudo-topological space.*

Proposition 4. 5. *Let (X_α, q_α) be a cnovergence space for each $\alpha \in \Lambda$.*

(a) If $((X_\alpha, q_\alpha))_{\alpha \in \Lambda}$ forms a pseudo-topologically coherent, $\rho(q_\alpha') = \rho(q_\alpha)$, $q_\alpha' \leqq q_\alpha$, then $((X_\alpha, q_\alpha'))_{\alpha \in \Lambda}$ also forms a pseudo-topologically coherent.

(b) If $((X_\alpha, q_\alpha))_{\alpha \in \Lambda}$ does not form a pseudo-topologically coherent, $\rho(q_\alpha') = \rho(q_\alpha)$, $q_\alpha' \geqq q_\alpha$, then $((X_\alpha, q_\alpha'))_{\alpha \in \Lambda}$ also fails to be a pseudo-topologically coherent

Proof. (a) $\rho(\Pi(X_\alpha, q_\alpha')) \leqq \rho(\Pi(X_\alpha, q_\alpha)) = \Pi(\rho(X_\alpha, q_\alpha)) = \Pi(\rho(X_\alpha, q_\alpha'))$

and by proposition 3. 3,

$$\rho(\Pi(X_\alpha, q_\alpha')) \geqq \Pi(\rho(X_\alpha, q_\alpha')).$$

Thus

$$\rho(\Pi(X_\alpha, q_\alpha')) = \Pi(\rho(X_\alpha, q_\alpha')),$$

that is, $((X_\alpha, q_\alpha'))_{\alpha \in \Lambda}$ forms a pseudo-topologically coherent.

(b) By assumption,

$$\Pi(\rho(X_\alpha, q_\alpha')) = \Pi(\rho(X_\alpha, q_\alpha)) \leqq \rho(\Pi(X_\alpha, q_\alpha)) \leqq \rho(\Pi(X_\alpha, q_\alpha')),$$

that is,

$$\Pi(\rho(X_\alpha, q_\alpha')) \neq \rho(\Pi(X_\alpha, q_\alpha')).$$

Thus $((X_\alpha, q_\alpha'))_{\alpha \in \Lambda}$ fails to be a pseudo-topologically coherent.

Proposition 4. 6. *If (X_α, q_α) is a convergence space for each $\alpha \in \Lambda$, $\Lambda' \subset \Lambda$ and*

$$\prod_{\alpha \in \Lambda - \Lambda'} X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta \geqq \rho(\prod_{\alpha \in \Lambda} X_\alpha),$$

then $(X_\alpha)_{\alpha \in \Lambda}$ is a pseudo-topologically coherent if and only if $(X_\alpha, \rho X_\beta)_{\alpha \in \Lambda - \Lambda', \beta \in \Lambda'}$ is a pseudo-topologically coherent.

Proof. If $(X_\alpha)_{\alpha \in \Lambda}$ is a pseudo-topologically coherent, then

$$\rho(\prod_{\alpha \in \Lambda - \Lambda'} X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta) \leqq \rho(\prod_{\alpha \in \Lambda} X_\alpha) = \prod_{\alpha \in \Lambda} \rho X_\alpha = \prod_{\alpha \in \Lambda - \Lambda'} \rho X_\alpha \times \prod_{\beta \in \Lambda'} \rho(\rho X_\beta).$$

Since

$$\prod_{\alpha \in \Lambda - \Lambda'} X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta \geqq \rho(\prod_{\alpha \in \Lambda} X_\alpha),$$

$$\rho(\prod_{\alpha \in \Lambda - \Lambda'} X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta) \geqq \rho(\rho \Pi X_\alpha) = \rho \Pi X_\alpha = \Pi \rho X_\alpha$$

$$= \prod_{\alpha \in \Lambda - \Lambda'} \rho X_\alpha \times \prod_{\beta \in \Lambda'} \rho(\rho X_\beta).$$

Thus

$$\rho(\prod_{\alpha \in \Lambda - \Lambda'} X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta) = \prod_{\alpha \in \Lambda - \Lambda'} \rho X_\alpha \times \prod_{\beta \in \Lambda'} \rho(\rho X_\beta),$$

so $(X_\alpha, \rho X_\beta)_{\alpha \in \Lambda - \Lambda',\, \beta \in \Lambda'}$ is a pseudo-topologically coherent.

Conversely, *if* $(X_\alpha, \rho X_\beta)_{\alpha \in \Lambda - \Lambda',\, \beta \in \Lambda'}$ is a pseudo-topologically coherent, since

$$\rho\left(\prod_{\alpha \in \Lambda} X_\alpha\right) \leqq \prod_{\alpha \in \Lambda - \Lambda'} X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta,$$

$$\rho\left(\prod_{\alpha \in \Lambda} X_\alpha\right) = \rho\left(\rho\left(\prod_{\alpha \in \Lambda} X_\alpha\right)\right) \leqq \rho\left(\prod_{\alpha \in \Lambda - \Lambda'} X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta\right)$$

$$= \prod_{\alpha \in \Lambda - \Lambda'} \rho X_\alpha \times \prod_{\beta \in \Lambda'} \rho(\rho X_\beta) = \prod_{\alpha \in \Lambda - \Lambda'} \rho X_\alpha \times \prod_{\beta \in \Lambda'} \rho X_\beta$$

$$= \prod_{\alpha \in \Lambda} \rho X_\alpha$$

and by proposition 3. 3,

$$\rho(\Pi X_\alpha) \geqq \Pi\, \rho X_\alpha.$$

Thus $(X_\alpha)_{\alpha \in \Lambda}$ is a pseudo-topologically coherent.

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