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On Determining Experimental Points in Central Composite Designs

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ABSTRACT

By using the tool for finding influential cases in regression, we investigate the properties of three types of diagonal elements of the hat matrix in central composite designs. From these results, we determine the experimental points at axes and the number of replicates at the center and the other points so that the experimental points give approximately same influences to estimate the response surface.

1. Introduction

Consider the general regression model

$$\underset{\sim}{y}=X\underset{\sim}{\beta}+\underset{\sim}{e} \tag{1.1}$$

where, $\underset{\sim}{y}$ is the $n\times 1$ vector of observed responses, X is $n\times p'$ matrix, $\underset{\sim}{\beta}$ is $p'\times 1$ vector to be estimated and $\underset{\sim}{e}$ is an $n\times 1$ vector of random errors, with mean vector $\underset{\sim}{0}$ and covariance matrix $I\sigma^2$. In the central composite design of the second order model in 2 independent variables, a scalar response, y, is determined by

$$y=\beta_0+\beta_1x_1+\beta_2x_2+\beta_{11}x_1^2+\beta_{22}x_2^2+\beta_{12}x_1x_2+e \tag{1.2}$$

and the experimental points are illustrated in Figure 1.1. In general, the central composite design in k independent variables has 2^k factorial, $2k$ axial points and a center point. The center point may be replicated n_0 times and we shall allow the other points to be replicated r times.

In this paper we consider the design aspects of response surface experiments in which emphasis is on determination of the values of α and the number of replicates, n_0 and r,

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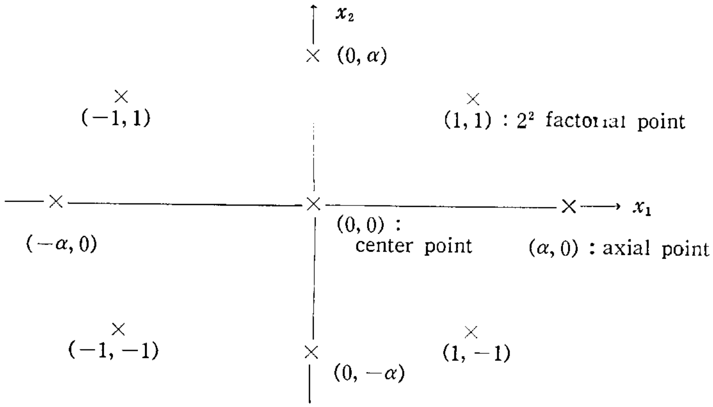


Figure 1.1. Experimental points in central composite design for $k=2$

for various values of k. Our purpose is to find designs which are insensitive to wild observations and to violation of the usual normal theory assumptions.

Recently, Cook(1977), Andrews and Pregibon(1978), Belsley, Kuh and Welsch(1980) and many other statisticians have provided the tools for finding influential cases in the linear model by using sample versions of the influence curve. In this paper, we will apply these results on regression for the central composite designs in the response surface analysis.

Cook(1977) proposed the Cook's distance D_i, as a measure of influence of the ith case, defined by

$$D_i = r_i^2 v_{ii} / \{p'(1-v_{ii})\}, \quad i=1, \cdots, n, \tag{1.3}$$

where p' is the number of parameters to be estimated including constant term, $r_i = \hat{e}_i / (\hat{\sigma}\sqrt{1-v_{ii}})$, $(i=1, \cdots, n)$ is the Studentized residual, $\hat{e}_i = y_i - \hat{y}_i$, $\hat{\sigma}^2 = \sum \hat{e}_i^2 / (n-p')$ and $v_{ii} = \underset{\sim}{x}_i'(X'X)^{-1}\underset{\sim}{x}_i$ $(i=1, \cdots, n)$ is the diagonal element of the hat matrix $V = X(X'X)^{-1}X'$.

For fixed p', the size of D_i will be determined by two different sources: the magnitude of r_i, a random variable reflecting the lack of fit of the model at the ith case, and the distance of th evector $\underset{\sim}{x}_i$ from the average of the other data vectors as reflected by v_{ii}. A large D_i may be determined by large r_i or by large v_{ii} or both. If we concentrate our attention on the problem of designing experimental points, however, what we can control between r_i and v_{ii} is only the value of v_{ii}. Therefore it is reasonable to find designs in which the values of v_{ii} are small and as close as possible.

Many authors have hinted the important role of v_{ii}. Box and Draper(1975) suggested that for a designed experiment to be insensitive to outliers, the v_{ii} should be small and approximately equal. On the other hand, we know that the eigenvalues of a hat matrix are either 0 or 1 and that the number of nonzero eigenvalues is equal to the rank of the matrix. In this case, $\text{rank}(V)=\text{rank}(X)=p'$, thus the average of the v_{ii} is p'/n. Hoaglin and Welsch (1978) suggested that a reasonable rule of thumb for large v_{ii} is $v_{ii}>2p'/n$. Huber (1981) pointed out that values $v_{ii}\leq 0.2$ appear to be safe, values between 0.2 and 0.5 are risky, and suggested that if we can control the design at all, we had better avoid values above 0.5. The purpose of this paper, from these points of view, is now focused on finding designs which satisfy the following conditions:

(a) v_{ii}'s are as close as possible.

(b) $v_{ii}\leq 0.2$, for all i. (1.4)

(c) Replicates are as small as possible.

In Section 2 the optimum designs for two independent variables satisfying the condition (a) in (1.4) are obtained. In Section 3 the optimum designs for general case are studied and we recommend those designs satisfying the conditions in (1.4). In Section 4, remark (a) are given.

2. OPTIMUM DESIGNS FOR TWO INDEPENDENT VARIABLES

2.1. Case when $r=1$

Throughout this paper, let n_0 be the number of replicates at the center point and r be that at the other points. Then the number of cases, n, of the model in (1.2) is given by $n=n_0+8r$ for a fixed r.

For a given n_0, the design matrix X in (1.2) is

$$X=\begin{pmatrix} 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\alpha & 0 & \alpha^2 & 0 & 0 \\ 1 & \alpha & 0 & \alpha^2 & 0 & 0 \\ 1 & 0 & -\alpha & 0 & \alpha^2 & 0 \\ 1 & 0 & \alpha & 0 & \alpha^2 & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} 4\ 2^2 \text{ factorial points} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n_0 \text{ center points} \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} 4 \text{ axial points} \end{matrix} \qquad (2.1)$$

$$(\quad x_1 \quad x_2 \quad x_1^2 \quad x_2^2 \quad x_1x_2)$$

and $X'X$ is

$$X'X=\left(\begin{array}{ccc:cc:c} n & 0 & 0 & f & f & 0 \\ 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 \\ \hdashline f & 0 & 0 & h & 4 & 0 \\ f & 0 & 0 & 4 & h & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right). \tag{2.2}$$

The inverse of $X'X$ in (1.2) is

$$(X'X)^{-1}=\left(\begin{array}{ccc:cc:c} s & 0 & 0 & -f/(nv) & -f/(nv) & 0 \\ 0 & 1/f & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/f & 0 & 0 & 0 \\ \hdashline -f/(nv) & 0 & 0 & t/w & -u/w & 0 \\ -f/(nv) & 0 & 0 & -u/w & t/w & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1/4 \end{array}\right) \tag{2.3}$$

where

$$n=n_0+8,\ f=2(\alpha^2+2),\ h=2(\alpha^4+2),\ t=h-f^2/n, \tag{2.4}$$
$$u=4-f^2/n,\ v=t+u,\ s=1/n+2f^2/(n^2v),\ w=v(t-u).$$

From the matrix in (2.3), we can derive the diagonal elements of $V=X(X'X)^{-1}X'$, which can be classified into three types as follows:

$$v_{ff}=1/(n_0+8)+1/4+1/(\alpha^2+2)+(2\alpha^2-4-n_0)^2/[(n_0+8)\{(n_0+4)\alpha^4-16\alpha^2+4(n_0+4)\}]$$
$$v_{cc}=1/(n_0+8)+4(\alpha^2+2)^2/[(n_0+8)\{(n_0+4)\alpha^4-16\alpha^2+4(n_0+4)\}]$$
$$v_{aa}=1/(n_0+8)+\alpha^2/\{2(\alpha^2+2)\}+\{(n_0+4)(n_0+6)\alpha^4-16(n_0+6)\alpha^2 \tag{2.5}$$
$$+2(n_0^2+12n_0+48)\}/[2(n_0+8)\{(n_0+4)\alpha^4-16\alpha^2+4(n_0+4)\}]$$

Note that the above v_{ff}, v_{cc} and v_{aa} are the diagonal elements of Vcorresponding to 2^2 factorial, center and axial points, respectively. Each v_{ii} falls in the range between 0 and 1. Actually if we let c be the number of rows in X that are exactly the same as $\underset{\sim}{x}_i$, including $\underset{\sim}{x}_i$, then

$$1/n\leq v_{ii}\leq 1/c. \tag{2.6}$$

Moreover, we can observe empirically the following.

As a function of α, v_{ff} is decreasing, while v_{aa} is increasing for any given n_0 less than 5. (2.7)

Since $\partial v_{cc}/\partial\alpha=64f\alpha(2-\alpha^2)/(n^2v^2)$ where f and v are those in (2.4), the following fact holds.

As a function of α, v_{cc} is concave downward with maximum value $1/n_0$ at $\alpha=\sqrt{2}$ for any given n_0. (2.8)

On the other hand, since $v_{ff}=v_{aa}=5/8$ at $\alpha=\sqrt{2}$ for all n_0, $\partial v_{ff}/\partial n_0=-A_1\alpha^4(\alpha^2-2)$ and $\partial v_{aa}/\partial n_0=-A_2(\alpha^2-2)$, where $A_1=A_1(\alpha,n_0)>0$ and $A_2=A_2(\alpha,n)>0$, and since $\partial v_{cc}/\partial n_0<0$ from v_{cc} in (2.5), we have the following.

As a function of n_0, v_{ii}'s are all decreasing for any give α except that v_{ff} and v_{aa} are constant at $\alpha=\sqrt{2}$ for all n_0. (2.9)

Note that v_{cc} is decreasing rapidly in n_0 and the decreasing rates of v_{ff} and v_{aa} are very little.

To find the values of α and n_0 which satisfy the condition (a) in (1.4) in the interesting region, $\{\alpha|0<\alpha<3\}$, first we solve the minimization problem in (2.10) with respect to α for a given n_0 empirically.

$$\textit{Minimize } |v_{ff}-v_{aa}|+|v_{aa}-v_{cc}|+v_{cc}-v_{ff}|. \qquad (2.10)$$

The solution of the above problem exists by the properties of v_{ii} in (2.6) through (2.9) and the results are as follows.

Table 2.1. Solutions for given n_0

n_0	α	min	v_{ff}	v_{cc}	v_{aa}
1	1.751	0.3608	0.5665	0.7468	0.7468
2	$\sqrt{2}$	0.2500	0.6250	0.5000	0.6250
3	$\sqrt{2}$	0.5833	0.6250	0.3333	0.6250

The solutions for $n_0>3$ are not necessary by the properties in (2.7) through (2.9). Now from Table 2.1, we can conclude that the optimum design is given when $\alpha=\sqrt{2}$ and $n_0=2$ for the case when $r=1$. Note that v_{ii}'s are greater than or equal to 0.5 in this case.

2.2 Cases when $r>1$

Now consider the optimum designs when the number of replicates, r, is greater thanor equal to 2. Similarly to Section 2.1, we can find the matrices $X'X$ and $(X'X)^{-1}$ of the same forms in (2.2) and (2.3) only except for $4r$ replacing 4. In this case, the expressions of t, v, s and w in (2.4) are also unchanged but those of n, f, h and u in (2.4) and v_{ii}'s in (2.5) are changed to

$$n=n_0+8r,\quad f=2r(\alpha^2+2),\quad h=2r(\alpha^4+2),\quad u=4r-f^2/n, \tag{2.11}$$

$$\begin{aligned} v_{ff}&=1/(n_0+8r)+1/(4r)+1/\{r(\alpha^2+2)\}\\ &\quad+(2r\alpha^2-4r-n_0)^2/[r(n_0+8r)\{(n_0+4r)\alpha^4-16r\alpha^2+4(n_0+4r)\}],\\ v_{cc}&=1/(n_0+8r)+4r(\alpha^2+2)^2/[(n_0+8r)\{(n_0+4r)\alpha^4-16r\alpha^2+4(n_0+4r)\}],\\ v_{aa}&=1/(n_0+8r)+\alpha^2/\{2r(\alpha^2+2)\}+\{(n_0+4r)(n_0+6r)\alpha^4-16r(n_0+6r)\alpha^2\\ &\quad+2(n_0^2+12n_0r+48r^2)\}/[2r(n_0+8r)\{(n_0+4r)\alpha^4-16r\alpha^2+4(n_0+4r)\}]. \end{aligned} \tag{2.12}$$

Investigating the equations in (2.12), we can find that the properties in (2.6) through (2.9) are valid for given r. To determine the values of α and n_0 for given r, we obtained Table 2.2 by solving the minimization problem in (2.10) with respect to α for given r and n_0 empirically.

Table 2.2. Solutions for given r and n_0 when $k=2$

r	n_0	α	min	v_{ff}	v_{cc}	v_{aa}
2	1	1.967	0.2177	0.2953	0.4042	0.4042
	2	1.751	0.1804	0.2832	0.3734	0.3734
	3*	$\sqrt{2}$*	0.0417	0.3125	0.3333	0.3125
	4	$\sqrt{2}$	0.1250	0.3125	0.2500	0.3125
	5	$\sqrt{2}$	0.2250	0.3125	0.2000	0.3125
3	3	1.751	0.1203	0.1888	0.2489	0.2489
	4	1.594	0.0760	0.1926	0.2306	0.2306
	5*	$\sqrt{2}$*	0.0667	0.2083	0.2000	0.2083
	6	$\sqrt{2}$	0.0833	0.2083	0.1667	0.2083
4	5	1.636	0.0675	0.1430	0.1768	0.1768
	6*	$\sqrt{2}$*	0.0208	0.1563	0.1667	0.1563
	7	$\sqrt{2}$	0.0268	0.1563	0.1429	0.1563
5	8*	$\sqrt{2}$*	0.0000	0.1250	0.1250	0.1250
6	10*	$\sqrt{2}$*	0.0083	0.1042	0.1000	0.1042

(*; optimum values of α and n_0 for given r)

Note that if we increase n_0 for fixed r, it becomes that $v_{cc}<v_{ff}$ and $v_{cc}<v_{aa}$, which is due to the properties in (2.7) through (2.9).

The optimum designs satisfying the condition (a) in (1.4) for given r are obtained from Tables 2.1 and 2.2 and they are summarized in Table 2.3.

Table 2.3. Optimum values of n_0 and α for given r

r	1	2	3	4	5	6
n_0	2	3	5	6	8	10
α	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$

3. Optimum designs for general case

3.1. Diagonal elements of hat matrix

In the central composite design of the second order model in k independent variables, the general regression model in (1.1) is represented explicitly by

$$y=\beta_0+\sum^{k}\beta_i x_i+\sum^{k}\beta_{ii}x_i^2+\sum_{i<j}^{k}\sum^{k}\beta_{ij}x_i x_j+e. \tag{3.1}$$

Thus the number of cases n is

$$n=n_0+2^k r+2kr \tag{3.2}$$

The $n\times p'$ design matrix X is of full column rank with $p'=1+2k+\binom{k}{2}$, and the symmetric matrix $X'X$ is

$$X'X=\left(\begin{array}{cccc|cccc|ccc}
n & & & & f & \cdots & \cdots & f & & & \\
 & f & & 0 & & & & & & & \\
 & & \ddots & & & & 0 & & & & \\
0 & & & f & & & & & & O & \\ \hline
f & & & & h & & & & & & \\
\vdots & & 0 & & & \ddots & r2^k & & & & \\
\vdots & & & & & r2^k & \ddots & & & & \\
f & & & & & & & h & & & \\ \hline
 & & & & & & & & r2^k & & O \\
 & & O & & & & & & & \ddots & \\
 & & & & & & & & O & & r2^k
\end{array}\right)\begin{array}{l} \}\ k+1 \\ \\ \\ \\ \}\ k \\ \\ \\ \\ \}\ \binom{k}{2} \\ \\ \\ \end{array} \tag{3.3}$$

Thus the inverse of $X'X$ is

$$
(X'X)^{-1}=\left(\begin{array}{cccc|ccc|ccc}
s & & & & -f/(nv) & \cdots & -f/(nv) & & & \\
 & 1/f & & 0 & & & & & & \\
 & & \ddots & & & 0 & & & & \\
 & 0 & & 1/f & & & & & & \\
\hline
-f/(nv) & & & & (v-u)/w & & -u/w & & 0 & \\
\vdots & & 0 & & & \ddots & & & & \\
-f/(nv) & & & & -u/w & & (v-u)/w & & & \\
\hline
 & & & & & & & 1/(r2^k) & & 0 \\
 & & & & 0 & & & & \ddots & \\
 & & & & & & & 0 & & 1/(r2^k)
\end{array}\right) \tag{3.4}
$$

where

$$
f=r(2\alpha^2+2^k),\quad h=r(2\alpha^4+2^k),\quad t=h-f^2/n,\quad u=r2^k-f^2/n, \tag{3.5}
$$
$$
v=t+(k-1)u,\quad s=1/n+kf^2/(n^2v),\quad w=v(t-u).
$$

From the inverse matrix in (3.4), we can derive the three types of diagonal elements v_{ff}, v_{cc} and v_{aa} of $V=X(X'X)^{-1}X'$ as follows:

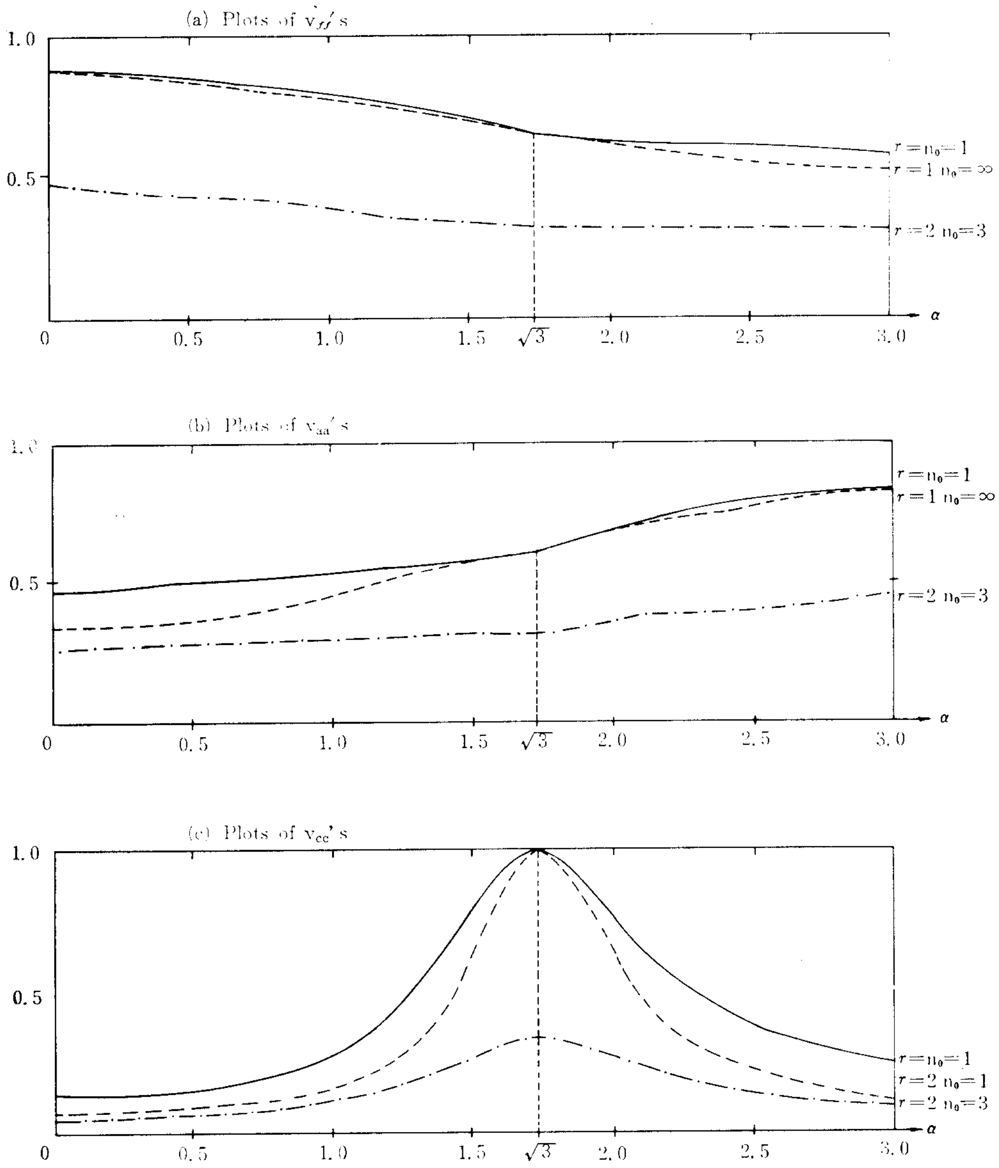
$$
\begin{aligned}
v_{ff}&=1/n+\binom{k}{2}/(r2^k)+k/f+k(f-n)^2/(n^2v),\\
v_{cc}&=1/n+kf^2/(n^2v),\\
v_{aa}&=1/n+\alpha^2/f+\{2rkf^2-4rnf\alpha^2+n^2(v-u)\}/(2rn^2v),
\end{aligned} \tag{3.6}
$$

where v_{ff}, v_{cc} and v_{aa} are those in Section 2.1. The property in (2.6) is valid for this case and the property in (2.7) is also valid for suitable k, r and n_0. Moreover, the property in (2.8) is valid for given k, r and n_0 except for $\alpha=\sqrt{k}$ replacing $\alpha=\sqrt{2}$, since $\partial v_{cc}/\partial\alpha=8r^2kf2^k\alpha(k-\alpha^2)/(n^2v^2)$, where f and v are those in (3.5). The property in (2.9) is also valid for given $k, r,$ and α except for $\alpha=\sqrt{k}$ replacing $\alpha=\sqrt{2}$. The reason is given below.

Clearly $\partial v_{cc}/\partial n_0<0$. $\partial v_{ff}/\partial n_0=-B_1\alpha^4(\alpha^2-k)^2$ and $\partial v_{aa}/\partial n_0=-B_2(\alpha^2-k)^2$, where $B_1=B_1(k,r,n_0,\alpha)>0$ and $B_2=B_2(k,r,n_0,\alpha)>0$. At $\alpha=\sqrt{k}$, $v_{ff}=\{(k^2-k)/(2^{k+1})+(k+1)/(2k+2^k)\}/r$ and $v_{aa}=\{1/2-1/(2k)+(k+1)/(2k+2^k)\}/r$, which are independent of n_0.

Figure 3.1 shows the functional properties of v_{ii}'s for the case when $k=3$.

Figure 3.1. Plots of v_{ii}'s as a function of α for $k=3$



3.2. Optimum designs for general case

For the cases when $k\geq 3$, the solutions of the minimization problem in (2.10) also exist by the properties of v_{ii} in Section 3.1 and we solve it with respect to α for given k, r and n_0 empirically, which are written in Table 3.1 for the case when $k=3$.

Table 3.1. Solutions for given r's and n_0's ($k=3$).

r	n_0	α	min	v_{ff}	v_{cc}	v_{aa}
1	1*	2.058*	0.1618	0.6289	0.7098	0.7098
	2	1.804	0.3090	0.6488	0.4943	0.6368
2	2	2.058	0.0809	0.3145	0.3549	0.3549
	3*	1.837*	0.0015	0.3222	0.3230	0.3230
	4	1.804	0.1545	0.3244	0.2472	0.3184
3	3	2.058	0.0539	0.2096	0.2366	0.2366
	4*	1.926*	0.0242	0.2116	0.2237	0.2237
	5	1.782	0.0376	0.2174	0.1987	0.2105
4	5	1.963	0.0249	0.1580	0.1705	0.1705
	6*	1.837*	0.0008	0.1611	0.1615	0.1615
	7	1.787	0.0422	0.1629	0.1418	0.1582
5	8*	1.877*	0.0109	0.1189	0.1189	0.1134
6	9*	1.837*	0.0005	0.1074	0.1077	0.1077

(*; optimum values of n_0 and α for given r)

Table 3.2. Optimum values of n_0 and α for given r and k

k	r	n_0	α	min	v_{ff}	v_{cc}	v_{aa}
2	1	2	$\sqrt{2}$	0.2500	0.6250	0.5000	0.6250
	2	3	$\sqrt{2}$	0.0417	0.3125	0.3333	0.3125
	3*	5*	$\sqrt{2}$*	0.0667	0.2083	0.2000	0.2083
	4	6	$\sqrt{2}$	0.0280	0.1563	0.1667	0.1563
	5	8	$\sqrt{2}$	0.0000	0.1250	0.1250	0.1250
	6	10	$\sqrt{2}$	0.0083	0.1042	0.1000	0.1042
3	1	1	2.058	0.1618	0.6289	0.7098	0.7098
	2	3	1.837	0.0015	0.3222	0.3230	0.3230
	3	4	1.926	0.0242	0.2116	0.2237	0.2237
	4*	6*	1.837*	0.0008	0.1611	0.1615	0.1615
	5	8	1.877	0.0109	0.1189	0.1189	0.1134
	6	9	1.837	0.0005	0.1074	0.1077	0.1077
4	1	1	1.661	0.1001	0.6160	0.6160	0.5660
	2	3	1.824	0.0362	0.2998	0.2998	0.2817
	3*	5*	2.058*	0.0100	0.1929	0.1979	0.1979
	4	7	2	0.0060	0.1458	0.1429	0.1458
	5	8	1.877	0.0109	0.1189	0.1189	0.1134
	6	10	2.508	0.0050	0.0965	0.0990	0.0990

k	r	n_0	α	min	v_{ff}	v_{cc}	v_{aa}
5	1	2	2.070	0.1378	0.4616	0.4616	0.5305
	2	3	1.907	0.0644	0.2340	0.2340	0.2662
	3*	5*	1.952*	0.0428	0.1554	0.1554	0.1768
	4	6	1.907	0.0322	0.1170	0.1170	0.1331
	5	8	1.934	0.0257	0.0934	0.0934	0.1062
6	1	2	2.215	0.3475	0.3320	0.5191	0.5191
	2	5	2.162	0.1714	0.1652	0.1652	0.2510
	3*	8*	2.211*	0.1142	0.1099	0.1099	0.1670
7	1	4	2.407	0.5188	0.2215	0.2280	0.4809
	2	8	2.407	0.2594	0.1108	0.1140	0.2405
	3*	11*	2.430*	0.1732	0.0738	0.0837	0.1604
8	1	6	2.640	0.6516	0.1427	0.1599	0.4686
	2	13	2.623	0.3256	0.0714	0.0735	0.2342
	3*	20*	2.617*	0.2170	0.0476	0.0477	0.1561
9	1	11	2.834	0.7458	0.0893	0.0893	0.4621
	2	22	2.834	0.3729	0.1446	0.0446	0.2310
	3*	33*	2.834*	0.2486	0.0298	0.0298	0.1540

(*; recommendable values of r, n_0 and α)

As shown in Table 3.1, we have found the optimum values of n_0 and α for given $k(>3)$ and r. The overall results for $2\leq k\leq 9$ are summarized in Table 3.2.

In Table 3.2, we recommended designs, the values of α and the numbers of replicates at the 2^k factorial and center points for given k, that require a minimum number of experimental points satisfying the conditions in (1.4). But the number n_0 of replicates at the center point is considerable for $k\geq 6$. Fortunately, we can reduce the number n_0 with small increase, indeed lessthan 0.01, of minimum value in (2.10) and without change in the condition of $v_{ii}\leq 0.2$. The following Table 3.3 shows the recommendable designs for various k, the number of independent variables, including the above facts.

Table 3.3. Recommendable values of r, n_0 and α for various k

k	2	3	4	5	6	7	8	9
r	3	4	3	3	3	3	3	3
n_0	5	6	5	5	3	4	4	5
α	$\sqrt{2}$	1.837	2.058	1.952	2.063	2.323	2.748	2.748

4. Concluding remrks

To what extent, a response surface design should satisfy some good properties is depending on circumstances. Our interest is retricted to the designs which are insensitive to wild ovservations and to violation of the normal theory assumptions, and which require a minimum number of experimental points. Thus, we have not considered those designs which guarantee the simplicity of the covariance matrix of an estimated β, etc.

When $k>5$, $v_{aa}>v_{ff}$ for all values of r, n_0 and α, which means that the axial points are always more influential than the 2^k factorial points. Therefore, finding the optimum designs by letting the number of replicates at the axial points differently from that at the 2^k factorial points is considered more desirable in the sense that this method would reduce the number of experiments. However, we leave it for a future study.

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