⟨**Research Paper**⟩

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On the E-optimality of different blocksize designs

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ABSTRACT

Constantine(1981) extended the results of Takeuchi(1961) by adding some new blocks to certain known E-optimal block designs. But they are confined to equal blocksize designs. In this paper we again generalize them to different blocksize case. By augmenting some known E-optimal block designs having blocks of equalsize with blocks of different sizes, additional E-optimal block designs are obtained.

1. Introduction

Let d be a block design with b blocks and v treatments. Then the information matrix of a block design d is defined as follows;

$$C(\mathrm{d})=\mathrm{diag}\ [r_1, r_2, \cdots, r_v]-N\ \mathrm{diag}\ [k_1^{-1}, k_2^{-1}, \cdots, k_b^{-1}]N' \qquad (1\cdot1)$$

where r_i's are replicates of treatments, k_i's are the sizes of blocks, and N is the incidence matrix whose entry n_{ij} gives the number of times treatment i occurs in block j.

We see that the matrix $C(d)$ is symmetric, non-negative definite and has zero row sums. A design d is said to be connected if all contrasts of treatment effects are estimable. It is known that a design d with v treatments is connected iff rank $(C(d))=v-1$. In this case we may assume the eigenvalues of $C(d)$ are

$$0=\mu_0(d)<\mu_1(d)\leq\mu_2(d)\leq\cdots\leq\mu_{v-1}(d)$$

Let $\Omega(v, b, k)$ denote the class of all connected block designs having v treatments

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arranged in b blocks of size k. A design d^* is said to be E-optimal in $\Omega(v, b, k)$ if among all designs in $\Omega(v, b, k)$, it minimizes the supremum of the variance of least square estimators for normalized treatment effect contrasts.

With the aid of Ehrenfeld (1955), we have a criterion for E-optimality; A design $d^* \epsilon \Omega(v, b, k)$ is E-optimal in $\Omega(v, b, k)$ iff

$$\mu_1(d^*) \geq \mu_1(d)$$

for any $d \epsilon \Omega(v, b, k)$, where $\mu_1(d)$ denotes the minimum positive eigenvalue of $C(d)$.

If d is a combined design of designs d_1 and d_2 under the same treatments, then clearly

$$C(d) = C(d_1) + C(d),$$

and

$$\mu_1(d) \geq \mu_1(d_i), \quad i=1, 2 \tag{1·2}$$

Using the property (Rao(1973))

$$\mu_1(d) = \inf_{\xi} (\xi' C(d) \xi),$$

where ξ is a normalized $v \times 1$ column vector with zero column sum, the following Lemmas can be shown in a similar way to Constantine(1981) and Jacroux (1982).

Lemma 1 : Let $C(d) = (c_{ij}(d))$ be the $v \times v$ information matrix of design d. Then

$$\mu_1(d) \leq \frac{c_{ii}(d) + c_{jj}(d) - 2c_{ij}(d)}{2}, \quad i \neq j.$$

Lemma 2 : Let $C(d)$ be as in Lemma 1, and M be a proper subset of the treatments $\{1, 2, \cdots, v\}$, say $M = \{1, 2, \cdots, m\}$, $1 \leq m < v$. Then

$$\mu_1(d) \leq \frac{v}{m(v-m)} \sum_{i, j \epsilon M} c_{ij}(d)$$

2. Main Retsults

Readers may refer to Raghavarao(1971) for the usual terminologies and notations on block design throughout the paper. Let $\Omega(v ; b_1, b_2 ; k_1, k_2)$ be the class of all connected block designs having v treatments arranged in b_i blocks of size k_i for $i=1, 2$.

The two theorems given in this section are extensions of the results proven in Constantine(1981) and Jacroux(1982).

Theorem 1 : Let $d_1^* \epsilon \Omega(v, b_1, k_1)$ be a BIB design and $d_2^* \epsilon \Omega(v, b_2, k_2)$ be an arbitrary binary design based on the same treatments with d_1^*. If $k_1 \geq k_2$ and $b_2 k_2 < v$, then the combined design $d^* = (d_1^*, d_2^*)$ is E-optimal in $\Omega(v ; b_1, b_2 ; k_1, k_2)$.

Proof : Since d_1^* is a BIB design,

$$r(d_1^*) = b_1 k_1 / v \text{ and } \lambda(d_1^*) = b_1 k_1 (k_1 - 1) / (v(v-1))$$

and we can see that

$$\mu_1(d_1^*)=\frac{r(d_1^*)(k_1-1)+\lambda(d_1^*)}{k_1}$$

So by (1·2), it follows that

$$\mu_1(d^*)\geq\frac{b_1(k_1-1)}{v-1} \tag{2·1}$$

Now let d be any design in $\Omega(v\,;b_1,b_2\,;k_1,k_2)$. Then d can be represented by $d=(d_1,d_2)$ for some $d_1\in\Omega(v,b_1,k_1)$ and $d_2\in\Omega(v,b_2,k_2)$. Hence $C(d)=C(d_1)+C(d_2)$, and so

$$r_i(d)=r_i(d_1)+r_i(d_2),\quad i=1,2,\cdots,v$$

and $\lambda_{ij}(d)=\lambda_{ij}(d_1)+\lambda_{ij}(d_2)$, $i\neq j$.

If we let $r_{i_0}(d)=\min_i r_i(d)$, then by the condition $\frac{b_2k_2}{v}<1$ we get $r_{i_0}(d)\leq r(d_1^*)$ since $r_i(d)$ is the replication of the i-th treatment. Then from Lemma 2 with $m=1$ and the condition $k_1\geq k_2$, it follows that

$$\mu_1(d)\leq\frac{b_1(k_1-1)}{v-1} \tag{2·2}$$

The result now follows from (2·1) and (2·2). ▨

Theorem 2 : Let d_1^* be a group divisible designs with parameters $v=mn$(m groups containing n treatments each), $b_1,k_1,\lambda_1,\lambda_2=\lambda_1+1$, and $d_2^*\in\Omega$ (v,b_2,k_2) be an arbitrary design based on the same treatments with d_1^*. If $k_1\geq k_2$ and $b_2k_2<v$-m, then the combined design $d^*=(d_1^*,d_2^*)$ is E-optimal in $\Omega(v\,;b_1,b_2\,;k_1,k_2)$.

Proof : Since d_1^* is a group divisible design,

$$r(d_1^*)=\frac{b_1k_1}{v},\quad \lambda_1(d_1^*)=\left[\frac{b_1k_1(k_1-1)}{v(v-1)}\right]$$

where [ ] denotes the integer part of the number, and

$$\mu_1(d_1^*)=\frac{r(d_1^*)(k_1-1)+\lambda_1(d_1^*)}{k_1}$$

Since $b_2k_2<v-m$, there exist at least 2 treatments i,j in d^* such that

$$c_{ii}(d^*)=c_{jj}(d^*)=\frac{r(d_1^*)(k_1-1)}{k_1},\quad c_{ij}(d^*)=-\frac{\lambda_1(d_1^*)}{k_1}$$

from (1, 1).

Then by Lemma 1,

$$\mu_1(d^*)\leq\frac{r(d_1^*)(k_1-1)+\lambda_1(d_1^*)}{k_1}$$

Since $\mu_1(d^*)\geq\mu_1(d_1^*)$ by (1·2), it follows that

$$\mu_1(d^*) = \frac{r(d_1^*)(k_1-1)+\lambda_1(d_1^*)}{k_1}$$

Now let $d \in \Omega(v; b_1, b_2; k_1, k_2)$ be arbitrary. Then $d=(d_1, d_2)$ for some $d_1 \in \Omega(v, b_1, k_1)$ and $d_2 \in \Omega(v, b_2, k_2)$.

Here we consider the two cases as follows;

(i) there exists some $r_1(d)$ such that $r_i(d) < r(d_1^*)$,

(ii) $r_i(d) \geq r(d_1^*)$ for $i=1, 2, \cdots, v$.

In case (i), by the condition $k_1 \geq k_2$ and $c_{ii}(d) = c_{ii}(d_1) + c_{ii}(d_2)$ we can show

$$c_{ii}(d) \leq \frac{(k_1-1)(r(d_1^*)-1)}{b_1}$$

Then by Lemma 2 and some algebra works analogous to those of Constantine(1981), we have

$$\mu_1(d) \leq \frac{v}{v-1} \quad \frac{(k_1-1)(r(d_1^*)-1)}{k_1} \leq \frac{r(d_1^*)(k_1-1)+\lambda_1(d_1^*)}{k_1}$$

Therefore $\mu_1(d) \leq \mu_1(d^*)$ in this case.

In a similar way, case (ii) is proved. ∎

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