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Another Look at Combined Intrablock and Interblock Estimation in Block Designs

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ABSTRACT

The relationships between combined estimators and generalized least squares estimators in block designs are reviewed. Here combined estimators mean the best linear combination of intrablock and interblock estimators. It is well known that only for balanced incomplete block designs the combined estimators of Yates and of the generalized least squares estimators give the same result. In this paper, a general form of the combined estimators for treatment effects is derived and it can be seen that such estimators are equivalent to the generalized least squares estimators.

1. Introduction

In a block design, if we assume fixed effect model, using ordinary least squares (OLS) method, the intrablock estimators of the parameters are obtained. Yates (1940) pointed out that if we assume the block effects are random variables (mixed model), a second set of estimators of treatment effects, called the interblock estimators, can be obtained. These two uncorrelated estimators, then, were combined to obtain an unbiased joint estimator with minimum variance. The original method of Yates for balanced incomplete block designs was to use weighted average of these two estimators. Consider a contrast $\phi=\sum c_i \tau_i$ in the treatments, where τ_i is the ith treatment effect in a block design. Let $\hat{\phi}_a=\sum c_i\hat{\tau}_{ai}$ and $\hat{\phi}_b=\sum c_i\hat{\tau}_{bi}$, where $\hat{\tau}_{ai}$ and $\hat{\tau}_{bi}$ are the estimators of ith treatment effect obtained by intrablock and interblock analyses, respectively, then the best combined estimator of ϕ has the form: $(w_1\hat{\phi}_a+w_2\hat{\phi}_b)/(w_1+w_2)$, where w_1 and w_2 are the inverses

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of variances of $\hat{\phi}_a$ and $\hat{\phi}_b$, respectively. Such combined estimation procedure should be optimum and there is no room to argue about this.

In the mixed model, however, it is well known that the best linear unbiased estimators can be obtained by generalized least squares (GLS) method. It is equivalent to Rao's maximum likelihood method (1947) under normality assumption, which is called Rao's combining method. It is well known that, in genenal, if the incomplete blocks are not balanced the methods of Yates and of Rao lead to different results (See Sprott (1956), Scheffé (1959), Kendall and Stuart (1966), or John (1971)). Actually in the literature Rao's combining method is more frequently used in the analysis of partially balanced incomplete block designs.

In this paper, a general form of the combined estimation formula for treatment effects is derived and it is shown that such estimators are equivalent to the GLS estimators.

2. Preliminaries

Let C be a symmetric $v\times v$ matrix whose rank is $v-1$ and $\mathbf{1}'C=0$, then $A=(C+\delta J)^{-1}$ is a generalized inverse of C and $G=A-\frac{1}{\delta v^2}J$ is the Moore-Penrose generalized inverse of C, where δ is an any nonzero constant, $\mathbf{1}$ is the $v\times 1$ vector with all elements unity and $J=\mathbf{1}\mathbf{1}'$, i.e., the $v\times v$ matrix all elements unity. The following fact will be important in our context: From $A(C+\delta J)=I$, $\delta vAJ=J$ since $CJ=0$, we get $AJ=\frac{1}{\delta v}J$ and $AC=I-\frac{1}{v}J$, where I is the $v\times v$ unit matrix.

Let there be v treatments. And suppose the ith treatment is replicated r_i times ($i=1, 2, \cdots, v$) in b blocks of k plots each, so $bk=\sum r_i=n$ (say). Let $N=(n_{ij})$, $i=1, 2, \cdots, v$; $j=1, 2, \cdots, b$, be the incidence matrix of the design, where n_{ij} equal to the number of times the ith treatment occurs in the jth block. The model assumed is

$$y_{ij}=\mu+\tau_i+\beta_j+\epsilon_{ij},\quad i=1, 2, \cdots, v;\ j=1, 2, \cdots, b, \tag{2.1}$$

where y_{ij} is the yield of the plot in the jth block to which the ith treatment is applied, μ is the overall mean effect, τ_i the effect of the ith treatment, β_j the effect of the jth block, and ϵ_{ij}'s are assumed to be independent variates with mean zero and variance σ^2. Let T_i be the total yield of all the plots having the ith treatment, B_j be the total yield of all the plots of the jth block and $\hat{\tau}_i$ be a solution for $\hat{\tau}_i$ in the normal equations. Further, denote the column vectors $(T_1, T_2, \cdots, T_v)'$, $(B_1, B_2, \cdots, B_b)'$, $(\tau_1, \tau_2, \cdots, \tau_v)'$, $(\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_v)'$ and $(\beta_1, \beta_2, \cdots, \beta_b)'$ by $\boldsymbol{T}$, $\boldsymbol{B}$, $\boldsymbol{\tau}$, $\hat{\boldsymbol{\tau}}$, and $\boldsymbol{\beta}$, respectively. The model (2.1) can be written in matrix form:

$$\boldsymbol{y}=\mathbf{1}\,\mu+X_1\,\boldsymbol{\tau}+X_2\,\boldsymbol{\beta}+\boldsymbol{\epsilon}, \tag{2.2}$$

where $\boldsymbol{y}$ is the $n\times 1$ yield vector. We may assume $\boldsymbol{y}=(\boldsymbol{y}_1', \boldsymbol{y}_2', \cdots, \boldsymbol{y}_b')'$ without loss of generality, where $\boldsymbol{y}_j$ is the vector of yields in the jth block, X_1 and X_2 are $n\times v$ and $n\times b$ matrices respectively and $\boldsymbol{\epsilon}$ is $n\times 1$ error vector with $E(\boldsymbol{\epsilon})=0$ and $V(\boldsymbol{\epsilon})=\sigma^2 I$.

We now have the following relationships between matrices in the model: $\mathbf{1}' X_1=\boldsymbol{r}'$, where $\boldsymbol{r}=(r_1, r_2, \cdots, r_v)'$, $\mathbf{1}' X_2=k\mathbf{1}_b'$, $X_1' X_1=R$, where $R=\mathrm{diag}(r_1, r_2, \cdots, r_v)$, $X_1' X_2=N$, $R\mathbf{1}=N\mathbf{1}=\boldsymbol{r}$, $N'\mathbf{1}=k\mathbf{1}$, $\mathbf{1}' R\mathbf{1}=\mathbf{1}' N\mathbf{1}=\sum r_i=n$, $X_1' \boldsymbol{y}=\boldsymbol{T}$, $X_2' \boldsymbol{y}=\boldsymbol{B}$ and $\mathbf{1}' \boldsymbol{y}=$total of all yields$=g$ (say).

3. Solutions of the normal equations

3.1 Intrablock analysis

Suppose the effects μ, $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ are assumed to be fixed constants, then $E(\boldsymbol{y})=\mathbf{1}\,\mu+X_1\,\boldsymbol{\tau}+X_2\,\boldsymbol{\beta}$, $V(\boldsymbol{y})=\sigma^2 I$. Using the OLS method, the reduced normal equations for the treatment effects is $C_a\hat{\boldsymbol{\tau}}_a=\boldsymbol{Q}_a{}^{(1)}$, where $C_a=R-\frac{1}{k}NN'$ and $\boldsymbol{Q}_a=\boldsymbol{T}-\frac{1}{k}N\boldsymbol{B}$. A solution of the normal equation is $\hat{\boldsymbol{\tau}}_a=A_a\boldsymbol{Q}_a=G_a\boldsymbol{Q}_a$, where $A_a=(C_a+\delta J)^{-1}$ and $G_a=A_a-\frac{1}{\delta v^2}J$. Note that $\mathbf{1}'\boldsymbol{Q}_a=0$ since $\mathbf{1}' C_a=0$, therefore $\mathbf{1}'\hat{\boldsymbol{\tau}}_a=0$ from the fact $\mathbf{1}' A_a=\frac{1}{\delta v}\mathbf{1}'$. In this case, $E(\boldsymbol{Q}_a)=C_a\boldsymbol{\tau}$, $V(\boldsymbol{Q}_a)=C_a\,\sigma^2$, $E(\hat{\boldsymbol{\tau}}_a)=A_aC_a\,\boldsymbol{\tau}=\left(1-\frac{1}{v}J\right)\boldsymbol{\tau}$, and $V(\hat{\boldsymbol{\tau}}_a)=A_a\,C_a\,A_a\,\sigma^2=G_a\,\sigma^2=G_a\frac{1}{w}$, where $w=\frac{1}{\sigma_2}$. Note that G_a is a singular matrix.

Thus $\hat{\boldsymbol{\tau}}_a$ is the unbiased estimator of $\left(I-\frac{1}{v}J\right)\boldsymbol{\tau}$. So, under the natural assumption $\mathbf{1}\boldsymbol{\tau}'=0$, $\hat{\boldsymbol{\tau}}_a$ is the unbiased estimator of $\boldsymbol{\tau}$.

Note: $\boldsymbol{Q}_a=\boldsymbol{T}-\frac{1}{k}N\boldsymbol{B}=\left(X_1'-\frac{1}{k}NX_2'\right)\boldsymbol{y}=\left(X_1'-\frac{1}{k}NX_2'\right)(\mathbf{1}\mu+X_1\boldsymbol{\tau}+X_2\boldsymbol{\beta}+\boldsymbol{\epsilon})=\left(T-\frac{1}{k}NN'\right)+\left(X_1-\frac{1}{k}NN'\right)\boldsymbol{\tau}+\left(X_1'-\frac{1}{k}NX_2'\right)\boldsymbol{\epsilon}$. Thus, the block effects vector $\boldsymbol{\beta}$ is vanished in $\boldsymbol{Q}_a$. So, even under mixed model $V(\hat{\boldsymbol{\tau}}_a)=G_a\,\sigma^2$.

3.2 Interblock analysis

Now suppose that block effects β_j and error terms ϵ_{ij} are random variables with means zero and variances $\sigma_\beta{}^2$ and σ^2, respectively, and all being uncorrelated. Under these assumptions, consider

$$X_2'\boldsymbol{y}=\boldsymbol{B}=k\mathbf{1}\,\mu+N'\,\boldsymbol{\tau}+kI\,\boldsymbol{\beta}+X_2'\boldsymbol{\epsilon}. \tag{3.2.1}$$

Then we have $E(\boldsymbol{B})=k\mathbf{1}\,\mu+N'\,\boldsymbol{\tau}$ and $V(\boldsymbol{B})I=k(k\sigma_\beta{}^2+\sigma^2)I=\frac{k}{w'}I$, where $w'=\frac{1}{\sigma^2+k\sigma_\beta{}^2}$

Applying the OLS method to the model (3.2.1), we get the reduced normal equation for $\hat{\tau}_b$: $C_b\hat{\tau}_b=Q_b$, where $C_b=NN'-\frac{1}{b}rr'$ and $Q_b=NB-\frac{1}{b}rg$. We get $\hat{\tau}_b=A_bQ_b=G_bQ_b$, where $A_b=(C_b+\delta J)^{-1}$ and $G_b=A_b-\frac{1}{\delta v^2}J$. Note that $1'\hat{\tau}_b=0$ because $1'A_bQ_b=\frac{1}{\delta v}1'Q_b=0$.

In this case, $E(Q_b)=C_b\tau$, $V(Q_b)=C_b\frac{k}{w'}$, $E(\hat{\tau}_b)=A_bC_b\tau=\left(I-\frac{1}{v}J\right)\tau=\tau$ under the assumption $1'\tau=0$, and $V(\hat{\tau}_b)=A_bC_bA_b\frac{k}{w'}=G_b\frac{k}{w'}$. Here we can see $E(\hat{\tau}_a)=E(\hat{\tau}_b)=\tau$ under the assumption $\sum\tau_i=0$.

This is the second set of estimators of treatment effects originated by Yates(1940), called the interblock estimators.

Intrablock estimator $\hat{\tau}_a$ and interblock estimator $\hat{\tau}_b$ are uncorrelated each other. That is

$$Cov(\hat{\tau}_a,\hat{\tau}_b)=\left(X_1'-\frac{1}{k}NX_2'\right)E[\epsilon(X_2\beta+\epsilon)']\left(NX_2'-\frac{1}{b}r1'X_2'\right)'$$
$$=\left(X_1'-\frac{1}{k}NX_2'\right)\left(X_2N'-\frac{1}{b}X_2 1 r'\right)\sigma^2=0$$

Note: If the blocksizes are different such that the jth blocksize is $k_j(j=1, 2, \cdots, b)$, then $V(B)=K(K\sigma_\beta^2+I\sigma^2)$, where $K=\text{diag}(k_1, k_2, \cdots, k_b)$, so the OLS method are not appropriate to apply. In such case, if we use GLS method, it is not Yates' interblock estimation in traditional sense. However, if we apply GLS method to such model, we reach to the same conclusion which will be derived in section 4.

3.3 Generalized least squares method

Under the same assumptions for the model (2.1) as in the previous subsection, we can see: $E(y)=1\mu+X_1\tau$ and $V(y)=\text{diag}(M, M, \cdots, M)=V$(say), where $M=\sigma^2I+\sigma_\beta^2J$. Note that $M^{-1}=\frac{1}{\sigma^2}I-\frac{\sigma_\beta^2}{\sigma^2(\sigma^2+k\sigma_\beta^2)}J=wI-\frac{1}{k}(w-w')J$, where $w=\frac{1}{\sigma^2}$ and $w'=\frac{1}{\sigma^2+k\sigma_\beta^2}$ and $V^{-1}=\text{diag}(M^{-1}, M^{-1}, \cdots, M^{-1})$, In practice, we have to estimate the variances σ^2 and σ_β^2 from the data. Hereafter we assume that σ^2 and σ_β^2 are known.

Applying GLS method we obtain the reduced normal equation for $\hat{\tau}_g$: $C_g\hat{\tau}_g=Q_g$, where $C_g=wR-\frac{1}{k}(w-w')NN'-w'\frac{rr'}{bk}$ and $Q_g=wT-\frac{1}{k}(w-w')NB-w'\frac{r}{bk}g$, where g is the total yield.

A solution of the normal equation is

$$\hat{\tau}_g=A_gQ_g=G_gQ_g, \tag{3.3.1}$$

where

$$A_g=(C_g+\delta J)^{-1} \text{ and } G_g=A_g-\frac{1}{\delta v^2}J.$$

It is interesting to note that C_g and Q_g can be expressed as: $C_g=wC_a+\frac{w'}{k}C_b$ and $Q_g=wQ_a+\frac{w'}{k}Q_b$.

4. Combining intrablock and interblock estimators

Let $\boldsymbol{t}'\boldsymbol{\tau}$ be a linear function of the elements of treatmnte effect vector $\boldsymbol{\tau}$. We want to estimate $\boldsymbol{t}'\boldsymbol{\tau}$ by sum of linear combinations of the elements of the intrablock estimator $\hat{\boldsymbol{\tau}}_a$ and interblock estimator $\hat{\boldsymbol{\tau}}_b$ such that $\boldsymbol{p}'\hat{\boldsymbol{\tau}}_a+\boldsymbol{q}'\hat{\boldsymbol{\tau}}_b=\boldsymbol{t}'\hat{\boldsymbol{\tau}}_c$(say) with minimum variance, where $\boldsymbol{p}'=(p_1, p_2, \cdots, p_v)$ and $\boldsymbol{q}'=(q_1, q_2, \cdots, q_v)$, and $\boldsymbol{t}'\hat{\boldsymbol{\tau}}_c$ is an unbiased estimator of $\boldsymbol{t}'\boldsymbol{\tau}$. Therefore $E(\boldsymbol{p}'\hat{\boldsymbol{\tau}}_a+\boldsymbol{q}'\hat{\boldsymbol{\tau}}_b)=\boldsymbol{t}'\boldsymbol{\tau}$, i.e., $(\boldsymbol{p}'+\boldsymbol{q}')\tau=\boldsymbol{t}'\boldsymbol{\tau}$ under assumption $\sum\tau_i=0$. This is true when

$$\boldsymbol{p}'+\boldsymbol{q}'=\boldsymbol{t}'. \tag{4.1}$$

Now $V(\boldsymbol{t}'\hat{\boldsymbol{\tau}}_c)=V(\boldsymbol{p}'\hat{\boldsymbol{\tau}}_a+\boldsymbol{q}'\hat{\boldsymbol{\tau}}_b)=\boldsymbol{p}'V(\hat{\boldsymbol{\tau}}_a)\boldsymbol{p}+\boldsymbol{q}'V(\hat{\boldsymbol{\tau}}_b)\boldsymbol{q}$ since $\hat{\boldsymbol{\tau}}_a$ is uncorrelated with $\hat{\boldsymbol{\tau}}_b$ and for $\boldsymbol{t}'\hat{\boldsymbol{\tau}}_c$ to be the best this variance must be a minimum, i.e., $\boldsymbol{p}'$ and $\boldsymbol{q}'$ are chosen to minimize $\boldsymbol{p}'V(\hat{\boldsymbol{\tau}}_a)\boldsymbol{p}+\boldsymbol{q}'V(\hat{\boldsymbol{\tau}}_b)\boldsymbol{q}$ subject to the condition $\boldsymbol{p}'+\boldsymbol{q}'=\boldsymbol{t}'$. Using $2\boldsymbol{\lambda}$ as a vector of Lagrange multipliers we, therefore, minimize

$$f(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\lambda})=\boldsymbol{p}'V(\hat{\boldsymbol{\tau}}_a)\boldsymbol{p}+\boldsymbol{q}'V(\hat{\boldsymbol{\tau}}_b)q-2\boldsymbol{\lambda}'(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{t})$$

with respect to elements of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{\lambda}$. Clearly $\frac{\partial f}{\partial \boldsymbol{\lambda}}=0$ gives (4.1) and $\frac{\partial f}{\partial \boldsymbol{p}}=0$ and $\frac{\partial f}{\partial \boldsymbol{q}}=0$ give

$$\boldsymbol{p}'V(\hat{\boldsymbol{\tau}}_a)=\boldsymbol{\lambda}' \text{ and } \boldsymbol{p}'V(\hat{\boldsymbol{\tau}}_b)=\boldsymbol{\lambda}', \tag{4.2}$$

respectively.

Since $V(\hat{\boldsymbol{\tau}}_a)=\frac{1}{w}G_a$, $V(\hat{\boldsymbol{\tau}}_b)=\frac{k}{w'}G_b$, $G_aC_a=I-\frac{1}{v}J$, and $G_bC_b=I-\frac{1}{v}J$,

$$\boldsymbol{p}'\left(I-\frac{1}{v}J\right)=w\boldsymbol{\lambda}'C_a, \text{ or } \boldsymbol{p}'=w\boldsymbol{\lambda}'C_a+\frac{1}{v}\boldsymbol{p}'J \tag{4.3}$$

$$\boldsymbol{q}'\left(I-\frac{1}{v}J\right)=\frac{w'}{k}\boldsymbol{\lambda}'C_b, \text{ or } \boldsymbol{q}'=\frac{w'}{k}\boldsymbol{\lambda}'C_b+\frac{1}{v}\boldsymbol{q}'J. \tag{4.4}$$

Hence,

$$\boldsymbol{t}'=\boldsymbol{\lambda}'\left(wC_a+\frac{w'}{k}C_b\right)+\frac{1}{v}\boldsymbol{t}'J, \text{ or } \boldsymbol{\lambda}'\left(wC_a+\frac{w'}{k}C_b\right)=\boldsymbol{t}'\left(I-\frac{1}{v}J\right) \tag{4.5}$$

Since $A_g\left(wC_a+\frac{w'}{k}C_b\right)=I-\frac{1}{v}J$, where $A_g=\left(wC_a+\frac{w'}{k}C_b+\delta J\right)^{-1}$ which is defined in (3.3.1), $\boldsymbol{\lambda}'\left(I-\frac{1}{v}J\right)=\boldsymbol{t}'\left(I-\frac{1}{v}J\right)A_g=\boldsymbol{t}'A_g-\frac{1}{\delta v^2}\boldsymbol{t}'J$, or

$$\boldsymbol{\lambda}'=\boldsymbol{t}'A_g+\frac{1}{v}\left(\boldsymbol{\lambda}'-\frac{1}{\delta v}\boldsymbol{t}'\right)J. \tag{4.6}$$

Hence, from (4.3), we get

$$\boldsymbol{p}'=w\boldsymbol{t}'A_gC_a+\frac{1}{v}\boldsymbol{p}'J$$

$$q' = \frac{w'}{k} t' A_g C_b + \frac{1}{v} q' J, \text{ since } JC_a = JC_b = 0.$$

Therefore

$$t' \hat{\tau}_c = t' A_g \left(wC_a \hat{\tau}_a + \frac{w'}{k} C_b \hat{\tau}_b \right). \tag{4.7}$$

Now suppose that t' takes the values u_i', the ith row of $v \times v$ unit matrix I. Then $u_i' \hat{\tau}_c = \tau_{ci}$, the ith element of $\hat{\tau}_c$. Thus by letting t' be, in turn, each row of I, the combined estimator can be written as:

$$\begin{aligned} \hat{\tau}_c &= A_g \left(wC_a \hat{\tau}_a + \frac{w'}{k} C_b \hat{\tau}_b \right) \\ &= A_g \left(w\boldsymbol{Q}_a + \frac{w'}{k} \boldsymbol{Q}_b \right) \\ &= A_g \left(w\boldsymbol{T} - \frac{1}{k} (w - w') N\boldsymbol{B} - \frac{w'}{bk} rg \right). \end{aligned} \tag{4.8}$$

Thus the expression of $\hat{\tau}_c$ is identical to that given in (3. 3. 1), i.e., the combined estimator $\hat{\tau}_c$ is equal to the GLS estimator $\hat{\tau}_g$.[2]

Now, we will show that for the balanced incomplete block design, our method and the of Yates give the same combined estimator.

In a balanced incomplete block design (v, r, k, b, λ), $C_a = rI - \frac{1}{k} NN' = \frac{\lambda v}{k} I - \frac{\lambda}{k} J$, $C_b = NN' - \frac{r^2}{b} J = (r - \lambda) I + \frac{\lambda b - r^2}{b} J$, $A_g^{-1} = wC_a + \frac{w'}{k} C_b + \delta J = \frac{1}{k} (w\lambda v + w'(r - \lambda)) I$, where $\delta = \frac{1}{bk} (wb(k + \lambda) + w'(b\lambda - r^2))$, so $A_g = k / (w\lambda v + w'(r - \lambda)) I$. Therefore, $A_g C_a = \lambda v / (w\lambda v + w'(r - \lambda)) I + a$ scalar multiple of J and $A_g C_b = k(r - \lambda) / (w\lambda v + w'(r - \lambda)) I + a$ scalar multiple of J. So we obtain

$$\begin{aligned} \hat{\tau}_c &= A_g \left(wC_a \hat{\tau}_a + \frac{w'}{k} C_b \hat{\tau}_b \right) \\ &= (w\lambda v + w'(r - \lambda))^{-1} (w\lambda v \hat{\tau}_a + w'(\lambda - \lambda) \hat{\tau}_b) \quad \text{since } \mathbf{1}' \hat{\tau}_a = \mathbf{1}' \hat{\tau}_b = 0. \end{aligned}$$

While, in Yates' method, letting $w_1 = \lambda v w / k$ and $w_2 = (v - \lambda) \frac{w'}{k}$,

$$\hat{\tau}_c = \frac{1}{w_1 + w_2} (w_1 \hat{\tau}_a + w_2 \hat{\tau}_b) = \frac{k}{w\lambda v + w'(r - \lambda)} \left(\frac{w\lambda v}{k} \hat{\tau}_a + \frac{w'(r - \lambda)}{k} \hat{\tau}_b \right).$$

Thus, in this case, the two methods give the same combined estimator, so we may understand that Yates' method is useful in balanced incomplete block designs.

Similary we may obtain convenient formulas for partially balanced incomplete block designs having Property A and more generally for multi-nested circulant incomplete block designs.

5. Comments

In practical situations, however, we do not know the variances of intrablock estimators and interblock estimators. Following Yates(1940), the method of estimating the variance components σ_β^2(block effect variance) and σ^2(error variance) by equating the mean square for blocks adjusted for treatments and the intrablock error mean square with their expectations which is known as the method of moments is usually recommended. However, it is known that the maximum likelihood or restricted maximum likelihood procedure (Patterson and Thompson, 1971) under normality assumption have more satisfactory theoretical properties than the method of moments(Harville, 1977), although the price is greater computational complexity. Recently, a computing method for maximum likelihood or restricted maximum likelihood estimators and adjusted means using existing computer program packages has been provided by Allen(1986).

The term 'mixed model' was introduced by Eisenhart(1947) to describe model useful in experiments where some effects, such as block or animal effects, can be thought of as random effects and other effects, for example treatments, are regarded as fixed. Since then, the intrablock analysis and interblock analysis(or recovery of interblock information) are simply analysis of fixed effect model and of mixed model, respectively. It seems to us that the terms 'intra- and inter-block' and 'combined estimators' are somewhat misleading. The purpose of this article is to emphasize this point of view.

Appendix

(1). Solution of the normal equation $C_a \hat{\tau}_a = \boldsymbol{Q}_a$

The solution of the normal equation for treatment effects in a block design is not unique. The solution $\hat{\tau}_a = A_a \boldsymbol{Q}_a$ is equvalent to the solution under restriction $\sum_i \hat{\tau}_{ai} = 0$. Each generalized inverse of C_a to solve the equation corresponds to a solution under one of thie restriction on $\hat{\tau}_a$. Thus $\hat{\tau}_a = A_a \boldsymbol{Q}_a$ is a solution among many possible ones. Traditionally, we solve the normal equation under a restriction for $\hat{\tau}_a$, therefore the numerical values of a solution $\hat{\tau}_a$ under one restriction should not be compared with other solutions. However, it is well known that the estimator of estimable function is unique under any restriction(See Searle, 1971).

(2). Tocher's derivation

If we follow Tocher (1952), the model for block designs which are equi-block-sized is

$$\boldsymbol{y}=X_1\boldsymbol{\mu}+X_2\boldsymbol{\beta}+\boldsymbol{\epsilon}, \quad (a.1)$$

where $\boldsymbol{\mu}=(\mu_1, \mu_2, \cdots, \mu_v)'$.

The intra-block estimator of $\boldsymbol{\mu}$ is: $\hat{\mu}_a=\Omega\left(\boldsymbol{T}-\frac{1}{k}N\boldsymbol{B}+\frac{\boldsymbol{r}}{bk}g\right)$ under assumption $\mathbf{1}'\boldsymbol{\beta}=0$ with $V(\hat{\boldsymbol{\mu}}_a)=\Omega\frac{1}{w}$, where $\Omega=\left(R-\frac{1}{k}NN'+\frac{1}{bk}\boldsymbol{r}\boldsymbol{r}'\right)^{-1}$ and interblock estimator of $\boldsymbol{\mu}_b$ is: $\hat{\boldsymbol{\mu}}_b=(NN')^{-1}N\boldsymbol{B}$ with $V(\hat{\boldsymbol{\mu}}_b)=(NN')^{-1}\frac{k}{w'}$. In this case, we also have $Cov(\hat{\mu}_a, \hat{\mu}_b)=0$. He achieved by applying the GLS method to the linear set up

$$E\begin{bmatrix}\hat{\boldsymbol{\mu}}_a\\ \hat{\boldsymbol{\mu}}_b\end{bmatrix}=\begin{bmatrix}I\\ I\end{bmatrix}\boldsymbol{\mu}, \qquad V\begin{bmatrix}\hat{\boldsymbol{\mu}}_a\\ \hat{\boldsymbol{\mu}}_b\end{bmatrix}=\begin{bmatrix}V(\hat{\boldsymbol{\mu}}_a) & 0\\ 0 & V(\hat{\boldsymbol{\mu}}_b)\end{bmatrix}=V, \text{ say.}$$

Then, the GLS normal equationfor $\hat{\mu}_c$, the combined estimator of $\boldsymbol{\mu}$, is

$$[I, \ I]V^{-1}\begin{bmatrix}I\\ I\end{bmatrix}\hat{\boldsymbol{\mu}}_c=V^{-1}[I, \ I]\begin{bmatrix}\hat{\boldsymbol{\mu}}_a\\ \hat{\boldsymbol{\mu}}_b\end{bmatrix}.$$

So, we get

$$\begin{aligned}\hat{\boldsymbol{\mu}}_c&=[V(\hat{\boldsymbol{\mu}}_a)^{-1}+V(\hat{\boldsymbol{\mu}}_b)^{-1}]^{-1}[V(\hat{\boldsymbol{\mu}}_a)^{-1}\hat{\boldsymbol{\mu}}_a+V(\hat{\boldsymbol{\mu}}_b)^{-1}\hat{\boldsymbol{\mu}}_b]\\ &=\left[w\Omega^{-1}+\frac{w'}{k}NN'\right]^{-1}\left[w\Omega^{-1}\hat{\boldsymbol{\mu}}_a+\frac{w'}{k}NN'\hat{\boldsymbol{\mu}}_b\right]\\ &=\left[R-\alpha NN'+\frac{1}{bk}\boldsymbol{r}\boldsymbol{r}'\right]^{-1}\left[\boldsymbol{T}-\alpha N\boldsymbol{B}+\frac{1}{bk}\boldsymbol{r}g\right], \quad (a.2)\end{aligned}$$

where $\alpha=(w-w')/kw$.

This is the solution obtained by Tocher (1952). However, our next conclusion is slightly different from his one.

Under the model (a.1) whose block effects are random variables, the GLS normal equation is

$$[R-\alpha NN']\hat{\boldsymbol{\mu}}_g=\boldsymbol{T}-\alpha N\boldsymbol{B} \quad (a.3)$$

Let $\Omega_c=\left(R-\alpha NN'+\frac{1}{bk}\boldsymbol{r}\boldsymbol{r}'\right)^{-1}$, then $\mathbf{1}'\Omega_c^{-1}=(2-\alpha k)\boldsymbol{r}'$. So $\boldsymbol{r}'\Omega_c=\frac{1}{(2-\alpha k)}\mathbf{1}'$.

Therefore

$$\begin{aligned}\boldsymbol{r}'\hat{\boldsymbol{\mu}}_c&=\frac{1}{(2-\alpha k)}\mathbf{1}'\left(\boldsymbol{T}-\alpha N\boldsymbol{B}+\frac{1}{bk}\boldsymbol{r}g\right)\\ &=\frac{1}{(2-\alpha k)}(g-\alpha kg+g)=g \quad (a.4)\end{aligned}$$

Now, from (a.2) and using (a.4), we get

$$[R-\alpha NN']\hat{\boldsymbol{\mu}}_c+\frac{1}{bk}\boldsymbol{r}g=\boldsymbol{T}-\alpha N\boldsymbol{B}+\frac{1}{bk}\boldsymbol{r}g.$$

That is, $[R-\alpha NN']\hat{\mu}_{c}=T-\alpha N\boldsymbol{B}$.

This shows that the solution (a. 2), the combined estimator, is equivalent to the solution of the GLS normal equation (a. 3). This result is equivalent to ours.

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