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Efficient Sequential Estimation in a Compound Poisson Process

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ABSTRACT

Sequential estimation of parameters in a compound Poisson process whose jump sizes are one-parameter exponential class random variables is discussed. Cramér-Rao type information inequality is used as an efficiency criterion. Unbiased estimators for certain parametric functions whose variances attain the lower bound are all characterized with the corresponding sampling plans.

1. Introduction

Compound Poisson process is encountered in many contexts such as queueing models and reliability theory, etc. In applying the process, the compounding distribution and/or its parameters are usually unknown, and must be estimated from the samples. Tucker (1963) estimated the moments of the compounding distribution. Simar (1976) suggested an iterative procedure for finding the maximum likelihood estimate of a compounding distribution.

We consider the problem of sequential estimation for a compound Poisson process in which the form of the compounding distribution is known but unknown parameters are involved. Determining a sequential estimation scheme involves the problem of defining and then finding optimal stopping rules or sampling plans. The most common criterion of optimality when working in unbiased estimation is given by the Cramér-Rao type information inequality. The equality is attained if and only if the sampling plan S, the

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estimator f, and the expected value $E(f)=g$ are 'efficient' in a sense to be specified.

The problem of characterizing efficient triples (S, f, g) has been studied by many authors: Girshick et al. (1946) and DeGroot (1959) for the case of binomial samples, Bhat and Kulkarni (1966) for the case of multinomial samples, Bai (1975) for finite irreducible Markov chains, Trybula (1982) and Adke and Manjunath (1984) for continuous time Markov processes, and Jang and Bai (1986) for Markov branching processes with im migration.

In this paper, assuming that the jump sizes are exponential class random variables, all the efficient triples are characterized in a compound Poisson process. In Section 2, the information inequality and the forms of efficient estimators are found. In Section 3, all the efficient triples are characterized. Some examples for well-known distributions of jump sizes are given in Section 4.

2. Information Inequality

Let $\{X(t),\ t\geq 0\}$ be a stochastic process represented by $X(t)=\sum_{1}^{N(t)} Y_i$, where $\{N(t),\ t\geq 0\}$ is a Poisson process with rate λ, Y_i's, $i=1, 2, \cdots$, are i.i.d. random variables, and $N(t)$ and Y_i's are independent. We assume that the p.d.f. of Y_1 is of exponential class type

$$f(y)=\exp[Q(\mu)\,T(y)+D(\mu)+K(y)], \tag{1}$$

where $Q(\mu)$ and $D(\mu)$ are real-valued functions of unknown parameter μ, and $T(y)$ and $K(y)$ are free of μ. We assume that $Q(\mu)$ and $D(\mu)$ are twice differentiable. Let $\Theta=\{(\lambda, \mu)\,;\ \lambda>0,\ -\infty<\mu<+\infty\}$. In what follows, we consider the sequential estimation of $\theta=(\lambda,\ \mu)\in\Theta$.

Suppose now that it is possible to observe the process continuously over a time interval $[0, \tau]$, where τ is a finite random stopping time. Then we have the likelihood (see Basawa and Rao (1980, p. 105))

$$L(\theta)=\lambda^{N(\tau)}e^{-\lambda\tau}\exp\left[Q(\mu)\sum_{1}^{N(\tau)} T(y_i)+N(\tau)D(\mu)+\sum_{1}^{N(\tau)} K(y_i)\right]. \tag{2}$$

Let $M(\tau)=\sum_{1}^{N(\tau)} T(y_i)$. We see from (2) that $W(\tau)=(\tau,\ N(\tau),\ M(\tau))$ is a sufficient statistic for θ.

We now derive the Cramér-Rao type information inequality. Suppose that $g(\theta)$ is a real-valued function of θ with finite partial derivatives for which there exists an unbiased estimator $f(W(\tau))$. Assume also that f is such that $E[f(W(\tau))]$ can be differentiated twice with respect to θ inside the integral sign. Let $g_\lambda'=\partial g/\partial\lambda$, $g_\mu'=\partial g/\partial\mu$. Also, let $Q'(\mu)=dQ(\mu)/d\mu$, $D'(\mu)=dD(\mu)/d\mu$, $Q''(\mu)=d^2Q(\mu)/d\mu^2$, and $D''(\mu)=d^2D(\mu)/d\mu^2$. Then we have the following.

$$E[(N(\tau)/\lambda-\tau)f]=g_\lambda'.$$
$$E[(Q'(\mu)\,M(\tau)+D'(\mu)\,N(\tau))f]=g_\mu'.$$
$$E[N(\tau)/\lambda-\tau]=E[Q'(\mu)\,M(\tau)+D'(\mu)\,N(\tau)]=0.$$
$$E[N(\tau)/\lambda-\tau]^2=E[N(\tau)]/\lambda^2.$$
$$E[Q'(\mu)\,M(\tau)+D'(\mu)\,N(\tau)]^2=-E[Q''(\mu)\,M(\tau)+D''(\mu)\,N(\tau)].$$
$$E[N(\tau)/\lambda-\tau][Q'(\mu)\,M(\tau)+D'(\mu)\,N(\tau)]=0.$$

Let $Z=\nabla\log L(\theta)$ where ∇ denotes the gradient operator. Then from the above results, we have

$$E[Z]=0, \tag{3}$$

$$\Sigma=E[ZZ^T]$$
$$=\begin{bmatrix} E[N(\tau)]/\lambda^2 & 0 \\ 0 & -E[Q''(\mu)\,M(\tau)+D''(\mu)\,N(\tau)] \end{bmatrix}. \tag{4}$$

Theorem 1. (Information inequality) Let $f(W(\tau))$ be an unbiased estimator of $g(\theta)$, $\theta=(\lambda,\ \mu)\in\Theta$, with finite variance. Under the assumptions of the above regularity conditions, we have

$$\mathrm{Var}(f)\geq\nabla g^T\,\Sigma^{-1}\,\nabla g. \tag{5}$$

Equality holds if and only if

$$f=g+\nabla g^T\,\Sigma^{-1}\,Z. \tag{6}$$

Proof. Let $A=\nabla g^T\,\Sigma^{-1}\,Z-(f-g(\theta))$. Then, $E[A]=0$, and

$$\mathrm{Var}(A)=\mathrm{Var}(f)-\nabla g^T\,\Sigma^{-1}\,\nabla g\geq0,$$

since $E[Zf]=\nabla g$. Thus the information inequality is

$$\mathrm{Var}(f)\geq\nabla g^T\,\Sigma^{-1}\,\nabla g$$
$$=\lambda^2 g_\lambda'^2/E[N(\tau)]-g_\mu'^2/E[Q''(\mu)\,M(\tau)+D''(\mu)\,N(\tau)].$$

Equality holds if and only if $\mathrm{Var}(A)=0$, which implies that, with probability 1,

$$f=g+\nabla g^T\,\Sigma^{-1}\,Z$$
$$=g(\theta)+a_1(\theta)\tau+a_2(\theta)\,N(\tau)+a_3(\theta)\,M(\tau), \tag{7}$$

where
$$a_1(\theta)=-\lambda^2 g_\lambda'/E[N(\tau)],$$
$$a_2(\theta)=\lambda g_\lambda'/E[N(\tau)]-D'(\mu)g_\mu'/K,$$
$$a_3(\theta)=-Q'(\mu)g_\mu'/K,$$

and $K=E[Q''(\mu)\, M(\tau)+D''(\mu)\, N(\tau)]$.

3. Efficient Triples

The notion of "efficient" estimators used in this paper is the one introduced by R.A. Fisher in connection with unbiased estimation. The following definitions are analogous to the ones used by DeGroot(1959), Bai(1975) and other authors.

Definition 1. For a given sampling plan S, a nonconstant estimator f is said to be efficient for $g(\theta)=E[f]$ at θ^* if equality holds in (5) when $\theta=\theta^*$. An unbiased estimator f is said to be efficient for $g(\theta)$ if it is efficient at all $\theta\in\Theta$, and then g is said to be efficiently estimable. A sampling plan S is efficient if it admits at least one efficiently estimable parametric function. A tripie (S, f, g) is then called an efficient triple.

Definition 2. A sampling plan S with stopping time τ is said to be closed if $P\{\tau<+\infty\}=1$ for all $\theta\in\Theta$.

We now characterize the efficient triples. In characterizing efficient triples (S, f, g), we consider parametric function g such that $\theta_1=\theta_2$ whenever $g(\theta_1)=g(\theta_2)$, $\theta_1, \theta_2\in\Theta$. Two distinct values of θ_1 and θ_2, $\theta_1, \theta_2\in\Theta$, are said to be equivalent with respect to g if $g(\theta_1)=g(\theta_2)$.

Theorem 2. Let S be a given sampling plan for which there exists a nonconstant estimator f efficient for some parametric function $g(\theta)$ at least at two values of θ that are not equivalent with respect to $g(\theta)$. Then there exist constants c_1, c_2, c_3, not all zero, and a constant $d(\neq 0)$ such that, with probability 1,

$$c_1\tau+c_2 N(\tau)+c_3 M(\tau)=d. \qquad (8)$$

Proof. From (7), f is efficient if and only if it can be written as

$$\begin{aligned} f&=g+\nabla g^T \Sigma^{-1} Z \\ &=g(\theta)+a_1(\theta)\tau+a_2(\theta)\, N(\tau)+a_3(\theta)\, M(\tau) \end{aligned}$$

with probability 1. Suppose f is efficient at θ_1 and θ_2 which are not equivalent with respect to $g(\theta)$. Then,

$$\begin{aligned} f&=g(\theta_1)+a_1(\theta_1)\tau+a_2(\theta_1)\, N(\tau)+a_3(\theta_1)\, M(\tau) \\ &=g(\theta_2)+a_1(\theta_2)\tau+a_2(\theta_2)\, N(\tau)+a_3(\theta_2)\, M(\tau). \end{aligned}$$

Hence,

$$c_1\tau+c_2 N(\tau)+c_3 M(\tau)=d,$$

where $c_i = a_i(\theta_1) - a_i(\theta_2)$, $i=1, 2, 3$, and $d = g(\theta_2) - g(\theta_1)$. The coefficients can not vanish simultaneously since θ_1 and θ_2 are not equivalent with respect to $g(\theta)$. Thus we have c_1, c_2, c_3, not all zero, and a constant $d(\neq 0)$ satisfying (8). ▨

Theorem 2 shows that if (8) is not satisfied, f can not be efficient at two or more values of θ.

Let $Z(t)$ be a function of the observations $[X(u),\ 0\leq u\leq t]$, $0\leq t<\infty$, and c be a real number. Then we write $S[Z(\tau);\ c]$ to denote the sampling plan according to which the process is observed until τ specified by

$$\tau = \inf[t>0;\ Z(t)=c].$$

Then, from (8), an efficient sampling plan S must be of the type

$$S[c_1\tau + c_2 N(\tau) + c_3 M(\tau);\ d], \tag{9}$$

where $d\neq 0$.

The following lemmas are useful to prove the closedness of the sampling plans.

Lemma 1. (see Jang and Bai(1986)) Let $\{N_i(t),\ t\geq 0\}$, $i=1, \cdots, m$, be independent Poisson processes with intensities α_i, respectively, and $Z(t)=\sum_1^m v_i N_i(t)$, where v_i, $i=1, \cdots, m$, are constants. Then, for any constants $\eta>0$ and v_0 with

$$\eta v_0 + \sum_1^m \alpha_i(\exp(\eta v_i) - 1) < 0, \tag{10}$$

and any $c>0$, we have

$$P\{\max_{t\geq 0}(Z(t) + v_0 t) < c\} > 0. \tag{11}$$

Note that for given v_i's, we can always choose α_i's satisfying (10).

Lemma 2. Let $\{N_i(t),\ t\geq 0\}$, $i=1, \cdots, m$, $m\geq 2$, be independent Poisson processes with intensities α_i, respectively and $\tau = \inf\left[t>0;\ v_0 t + \sum_1^m v_i N_i(t) = d\right]$, where $d>0$. Then

$$P\{\tau < +\infty\} < 1 \tag{12}$$

if the coefficients v_0 and v_i's, $i=1, \cdots, m$, satisfy one of the following five conditions:

(i) $v_0=0$ and $v_i\geq 0$ with at least two $v_i>0$,

(ii) $v_0=0$ and $v_i v_j<0$ for some i, $j (i\neq j)$,

(iii) $v_0>0$ and $v_i\geq 0$ with at least one $v_i>0$,

(iv) $v_0<0$ with at least one $v_i>0$,

(v) $v_0>0$ with at least one $v_i<0$,

Proof. (i) : Let $v_0=0$ and $v_i \geq 0$ with at least two $v_i>0$. Assume without loss of generality that $0<v_1<v_2$. Since v_1 and v_2 are both positive we can find non-negative integers k and s such that

$$v_1 k+v_2 s<d \text{ and } v_1 k+v_2(s+1)>d.$$

Consider the event A defined by

$$A=\{N_1(t)=k,\ N_2(t)=s,\ N_2(t+u)=s+1,\ N_j(t+u)=0 \text{ for all } 3\leq j\leq m,\ 0\leq t,\ u<\infty\}$$

Since $N_1(t)$ and $N_2(t)$ are independent Poisson processes and $\lim_{t\to\infty} N_i(t)=\infty$, we have $P\{A\}>0$, which implies (12).

(ii) : Let $v_0=0$ and $v_i v_j<0$ for some i, $j(i\neq j)$. From Lemma 1, for some $\eta>0$, and v_0' satisfying (10), we then have

$$P\left\{\sum_1^m v_i N_i(t)\neq d\right\}$$

$$\geq P\left\{\max_{t\geq 0} \sum_1^m v_i N_i(t)<d\right\}$$

$$\geq P\left\{\max_{t\geq 0}\left[v_0' t+\sum_1^m v_i N_i(t)\right]<d\right\}>0.$$

(iii) : Let $v_0>0$ and $v_i\geq 0$ with at least one $v_i>0$. Without loss of generality, assume that $v_1>0$. Then

$$P\left\{v_0 t+\sum_1^m v_i N_i(t)\neq d\right\}$$

$$> P\{N_i(t)=0,\ N_1(s)=1,\ 0\leq t\leq s,\ 1\leq i\leq m\}>0,$$

where $s=(d-w)/v_0$ and $0<w<v_1$.

(iv) : Let $v_0<0$ with at least one $v_i>0$. From Lemma 1, we see that, for any constant η and v_0 satisfying (10),

$$P\left\{v_0 t+\sum_1^m v_i N_i(t)\neq d\right\}$$

$$\geq P\left\{\max_{t\geq 0}\left[v_0 t+\sum_1^m v_i N_i(t)\right]<d\right\}>0.$$

(v) : The case where $v_0>0$ with at least one $v_i<0$, can be handled with arguments similar to the case (iv). ▨

Let d_1 be a positive real number, d_2, d_3, and d_4 be positive integers. The following theorem gives closed efficient sampling plans. If, from (9) $c_2\neq 0$ and $T(Y_1)$ in $M(\tau)$ is a continuous random variable, any sampling plan is clearly not closed. Thus we only consider the case where $T(Y_1)$ is a discrete random variable.

Theorem 3. Let $S_1=S[\tau\ ;\ d_1]$, $S_2=S[N(\tau)\ ;\ d_2]$, $S_3=S[M(\tau)/k\ ;\ d_3]$, and $S_4=S[N(\tau)-M(\tau)/k\ ;\ d_4]$. Then S_1 and S_2 are closed efficient sampling plans. S_3 and S_4 are closed efficient sampling plans if and only if

$$P\{T(Y_1)=k\}=1-P\{T(Y_1)=0\}>0, \tag{13}$$

for some positive constant k.

Proof. From (9), we can easily see that S_i's, $i=1, \cdots, 4$, are efficient. We now show the closedness of S_i's. Without loss of generality, assume that $d>0$. If $c_1>0$, $c_2=c_3=0$, we get S_1, which is closed since $d/c_1=d_1$ is always finite. If we set $c_1=c_3=0$ and $c_2>0$ for which $d/c_2=d_2$, we get S_2. The closedness of S_2 is clear since $\lim_{t\to\infty} N(t)=\infty$. Clearly, if d_2 is not an integer, the sampling plan is not closed. Also if we set $c_1=c_2=0$ and $c_3>0$ for which $d/c_3=kd_3$, under (13), we get S_3. The closedness of S_3 is clear since $\lim_{t\to\infty} M(t)/k=\infty$. If d_3 is not an integer, the sampling plan is not closed. We now show that S_3 is not closed if (13) is not satisfied.

Suppose that $T(y_i)$ is a discrete random variable having m possible values $v_1, \cdots, v_m$, $m\geq 2$. Let $N_j(t)$, $j=1, \cdots, m$, be the number of times events $\{T(y_i)=v_j\}$ occurs, $i=1, \cdots, N(t)$. Then $N(t)=\sum_1^m N_j(t)$ and $M(t)=\sum_1^{N(t)} T(y_i)=\sum_1^m v_j N_j(t)$. It can be shown that $N_j(t)$'s, $j=1, \cdots, m$, are independent Poisson processes with rates $\lambda_j=P\{T(y_i)=v_j\}$. Then, the sampling plan is

$$S[M(\tau)/k\ ;\ d_3]=S\Big[\sum_1^m (v_j/k)\ N_j(\tau)\ ;\ d_3\Big].$$

Thus, from (i) and (ii) of Lemma 2, S_3 is not closed. That is, if (13) is not satisfied, S_3 is not closed. Similarly it can be shown that S_4 is a closed efficient sampling plan if and only if (13) holds. ▨

Theorem 3 shows that S_1, S_2, S_3, and S_4 are closed efficient sampling plans. We now show that there exists no other closed efficient sampling plans.

Theorem 4. Let S be the class of all closed efficient sampling plans. Then $S=\{S_1, S_2, S_3, S_4\}$.

Proof. We now demonstrate no other combinations of c_1, c_2, and c_3 in (9) would lead to closed sampling plans. The efficient sampling plan of the type (9) can be written

$$S[c_1\tau+c_2 N(\tau)+c_3 M(\tau)\ ;\ d]$$

$$=S\left[c_1\tau+\sum_1^m (c_2+c_3 v_j)\, N_j(\tau)\ ;\ d\right].$$

If $c_1\neq 0$ and at least one coefficient of $N_i(t)$ is not zero, the sampling plan is not closed from (iii)$\sim$(v) of Lemma 2. If $c_1=0$ and at least two coefficients of $N_i(t)$ is not zero, then the sampling plan is not closed from (i) and (ii) of Lemma 2. Thus we can see that the only closed efficient sampling plans are S_1, S_2, S_3, and S_4.

The following theorem gives all efficient triples. Let a_i's, $i=0, 1, 2, \cdots$, be known constants.

Theorem 5. Under S_i, the parametric function $g_i(\theta)$ is efficiently estimable by f_i, $i=1, \cdots, 4$, with the additional constraint (13) for the last two cases, where

$$g_1=a_0+d_1\lambda[a_1-a_2 D'(\mu)/Q'(\mu)], \qquad f_1=a_0+a_1 N(\tau)+a_2 M(\tau),$$
$$g_2=a_0+d_2[a_1/\lambda-a_2 D'(\mu)/Q'(\mu)], \qquad f_2=a_0+a_1\tau+a_2 M(\tau),$$
$$g_3=a_0-kd_3 Q'(\mu)[a_1/\lambda+a_2]/D'(\mu), \qquad f_3=a_0+a_1\tau+a_2 N(\tau),$$
$$g_4=a_0+d_4 kQ'(\mu)[a_1/\lambda+a_2]/(kQ'(\mu)+D'(\mu)), \qquad f_4=a_0+a_1\tau+a_2 N(\tau).$$

Proof. We will show that g_1 is efficiently estimable by f_1 under S_1. Since the coefficients of r.v.'s in (7), must be constant for all 0, we obtain a system of equations by substituting d_1 for τ :

$$g-d_1\lambda^2 g_\lambda'/E[N(\tau)]=a_0, \tag{14}$$

$$\lambda g_\lambda'/E[N(\tau)]-D'(\mu)g_\mu'/K=a_1, \tag{15}$$

and

$$-Q'(\mu)g_\mu'/K=a_2. \tag{16}$$

From (14) through (16), we get

$$g=a_0+d_1\lambda[a_1-a_2 D'(\mu)/Q'(\mu)],$$

and

$$f=a_0+a_1 N(\tau)+a_2 M(\tau).$$

Similarly, it can be shown that g_i is efficiently estimable by f_i under S_i, $i=2, 3, 4$. ▨

4. Examples

We now give several examples applied to some of the well-known jump size distributions: Bernoulli, negative binomial, normal, and exponential distributions.

Example 1. (Bernoulli distribution with parameter p):

In this case $Q(p)=\log(p/(1-p))$, $D(p)=\log(1-p)$, $M(\tau)=X(\tau)$, and efficient triples are as follows:

(1) $S_1=S[\tau\ ;\ d_1]$,
$g_1=a_0+d_1\lambda[a_1+a_2 p]$,
$f_1=a_0+a_1 N(\tau)+a_2 X(\tau)$.

(2) $S_2=S[N(\tau)\ ;\ d_2]$,
$g_2=a_0+d_2[a_1/\lambda+a_2 p]$,
$f_2=a_0+a_1\tau+a_2 X(\tau)$.

(3) $S_3=S[X(\tau)\ ;\ d_3]$,
$g_3=a_0+d_3[a_1/\lambda+a_2]/p$,
$f_3=a_0+a_1\tau+a_2 N(\tau)$.

(4) $S_4=S[N(\tau)-X(\tau)\ ;\ d_4]$,
$g_4=a_0+d_4[a_1/\lambda+a_2]/(1-p)$,
$f_4=a_0+a_1\tau+a_2 N(\tau)$.

Remark 1.

(a) λ is efficiently estimable by $N(\tau)/d_1$ under $S[\tau\ ;\ d_1]$.

(b) $1/\lambda$ and p are efficiently estimable by τ/d_2 and $X(\tau)/d_2$, respectively, under $S[N(\tau)\ ;\ d_2]$.

(c) $1/p$ is efficiently estimable by $N(\tau)/d_3$ under $S[X(\tau)\ ;\ d_3]$.

(d) $1/(1-p)$ is efficiently estimable by $N(\tau)/d_4$ under $S[N(\tau)-X(\tau)\ ;\ d_4]$.

Example 2. (Negative binomial distribution $NB(r, p)$ with parameters r and p):
We assume that r is a known integer. We then have $Q(p)=\log(1-p)$, $D(p)=r\log(p)$, and $M(\tau)=X(\tau)$. Let $m=q/p$, where $q=1-p$. Then efficient triples are as follows:

(1) $S_1=S[\tau\ ;\ d_1]$,
$g_1=a_0+d_1\lambda[a_1+a_2 rm]$,
$f_1=a_0+a_1 N(\tau)+a_2 X(\tau)$.

(2) $S_2=S[N(\tau)\ ;\ d_2]$,
$g_2=a_0+d_2[a_1/\lambda+a_2 rm]$,
$f_2=a_0+a_1\tau+a_2 X(\tau)$.

(3) $S_3=S[X(\tau)\ ;\ d_3]$,
$g_3=a_0+d_3[a_1/\lambda+a_2]/rm$,
$f_3=a_0+a_1\tau+a_2 N(\tau)$

(4) $S_4=S[N(\tau)-X(\tau)\ ;\ d_4]$,
$g_4=a_0+d_4[a_1/\lambda+a_2]/(1-rm)$,
$f_4=a_0+a_1\tau+a_2 N(\tau)$.

Remark 2.

(a) If $r=1$, we have efficient triples when jump sizes have geometric distribution with parameter p.

(b) λ is efficiently estimable by $N(\tau)/d_1$ under $S[\tau\ ;\ d_1]$.

(c) $1/\lambda$ and rm are efficiently estimable by τ/d_2 and $X(\tau)/d_2$, respectively, under $S[N(\tau)\ ;\ d_2]$.

(d) $1/rm$ is efficiently estimable by $N(\tau)/d_3$ under $S[X(\tau)\ ;\ d_3]$.

(e) $1/(1-rm)$ is efficiently estimable by $N(\tau)/d_4$ under $S[N(\tau)-X(\tau)\ ;\ d_4]$.

Example 3. (Normal ditribution with mean μ and variance 1) :
We have $Q(\mu)=\mu$, $D(\mu)=-\mu^2$, and $M(\tau)=X(\tau)$. Efficient triples are as follows;

(1) $S_1=S[\tau\ ;\ d_1]$,

(2) $S_2=S[N(\tau)\ ;\ d_2]$,

$g_1=a_0+d_1\lambda[a_1+a_2\mu]$, $g_2=a_0+d_2[a_1/\lambda+a_2\mu]$,

$f_1=a_0+a_1N(\tau)+a_2X(\tau)$. $f_2=a_0+a_1\tau+a_2X(\tau)$.

Remark 3.

(a) λ is efficiently estimable by $N(\tau)/d_1$ under $S[\tau\ ;\ d_1]$.

(b) $1/\lambda$ and μ are efficiently estimable by τ/d_2 and $X(\tau)/d_2$, respectively, under $S[N(\tau)\ ;\ d_2]$.

Example 4. (Exponential distribution with parameter μ) :

We have $Q(\mu)=-\mu$, $D(\mu)=\log(\mu)$, and $M(\tau)=X(\tau)$. Efficient triples are as follows;

(1) $S_1=S[\tau\ ;\ d_1]$, (2) $S_2=S[N(\tau)\ ;\ d_2]$,

$g_1=a_0+d_1\lambda[a_1+a_1\mu]$, $g_2=a_0+d_2[a_1/\lambda+a_2\mu]$,

$f_1=a_0+a_1N(\tau)+a_2X(\tau)$. $f_2=a_0+a_1\tau+a_2X(\tau)$.

Remark 4.

(a) λ is efficiently estimable by $N(\tau)/d_1$ under $S[\tau\ ;\ d_1]$.

(b) $1/\lambda$ and μ are efficiently estimable by τ/d_2 and $X(\tau)/d_2$, respectively under $S[N(\tau)\ ;\ d_2]$.

References

(1) Adke, S.R. and Manjunath, S.M. (1984). Sequential Estimation for Continuous Time Finite Markov Processes. *Comm. Statist. Theory and Method.*, Vol. 13, No. 9, 1089~1106.

(2) Bai, D.S. (1975). Efficient Estimation of Transition Probabilities in a Markov Chain. *Ann. Statist.*, Vol. 3, No. 6, 1305~1317.

(3) Basawa, I.V. and Prakasa Rao, B.L.S. (1980). *Statistical Inference for Stochastic Processes.* Academic Press.

(4) Bhat, B.R. and Kulkarni, N.V. (1966). On Efficient Multinomial Estimation. *J. Roy. Statist. Soc. Ser. B.*, Vol. 28, 45~52.

(5) DeGroot, M.H. (1959). Unbiased Sequential Estimation for Binomial Populations. *Ann. Math. Statist.*, Vol. 30, 80~101.

(6) Girshick, M.A., Mosteller, F., and Savage, L.J. (1946). Unbiased Estimates for Certain Binomial Sampling Problems with Applications. *Ann. Math. Statist.*, Vol. 17, 13~23.

(7) Jang, J.S., and Bai, D.S. (1986). Efficient Sequential Estimation in Continuous Time Markov Branching Processes. *Comm. Statist. Theory and Method.*, Vol. 15, No. 5, 1681~1697.

(8) Simar, L. (1976). Maximum Likelihood Estimation of a Compound Poisson Process. *Ann. Statist.*, Vol. 4, 1200~1209.

(9) Tucker, H.G. (1963). An Estimate of the Compounding Distribution of a Compound Poisson Process, *Theory. Probab. Appl.*, Vol. 8, 195~200.