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—Research Paper—

Flexible Mixed Decomposition Method for Large Scale Linear Programs: —Integration of a Network of Process Models—

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Abstract

In combining dispersed optimization models, either primal or dual (or both) decomposition method is widely used as an organizing device. Interpreting the methods economically, the concepts of price- and resource-directive coordination are generally well accepted. Most of decomposition/integration methods utilize either primal information or dual information, not both, from subsystems, while some authors have developed mixed decomposition approaches employing two master problems dealing primal and dual proposals separately. In this paper a hybrid decomposition method is introduced, where one hybrid master problem utilizes the underlying relationships between primal and dual information from each subsystem. The suggested method is well justified with respect to the flexibility in information flow pattern choice (some prices and other quantities) and to the compatibility of subdivision's optimum to the systemwide optimum, that is often lacking in conventional decomposition methods such as Dantzig-Wolfe's. A numerical example is also presented to illustrate the suggested approach.

1. Introduction

In the last twenty five years since Dantzig and Wolfe [5] published the "Decomposition Principle" in 1960, numerous decomposition techniques to handle large-scale systems have been developed (see, for example, comprehensive survey articles by Geoffrion [7], Ruefli [14], Luna [12], and Gijsbrecht [8]). These works provide not only the optimal solution associated with large systems, but also help understand the problem of decentralized decision making. In fact, there exists a vast literature in this field under the headings such as large scale mathematical programming, decentralized decision making, decomposition/coordination, multi-level economic planning, or resource allocation.

The systems addressed to by these works are usually consist of a set of subsystems with their own divisional constraints, yet tied together via some linking resources (commonly called 'common resources' if these are inputs to all subdivisions, and to be called 'input-output' resources

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if these are inputs to some subdivisions but outputs from others). Without these linking constraints, each division can be solved individually. In the presence of the common constraints, however, a center (or a master) and some type of information flow structure is needed to maintain the common resource balances or the optimal allocation of common resources while allowing each of intradivision optimum. This can be done typically in three ways, namely,

1) price directive approach
2) resource directive approach
3) mixed approach.

The well known decomposition method by Dantzing and Wolfe [5] originally applied to *LP* problems with block-angular structures is a typical example of the price directive approach. Here, the master charges the subdivisions for the use of linking resources or subsidizes them for the production of linking resources. Given this price information, each subdivision finds and reports to the center its own optimal activity pattern in the form of linking resource requirements and the contribution to the central objective. With this new information the center sets new prices, which in turn is notified again back to subdivisions. The center attempts to harmonize the subdivisions in an iterative fashion to make intradivisional decisions which confirm to the overall system optimal. Thus, in this approach the price information is given from the center to subdivisions, while the quantity information is going opposite.

The second approach has the opposite information flows. The subdivision sets the prices, while the center sets the quantities. In other words, the center assigns the linking resources among subdivisions. Then subdivisions, from the optimal programs attainable under the given linking resource quotas, reports to the center the prices they are willing to pay for the use of resources or the prices they claim to receive for the output resources. The center improves the allocation of quota efficiently by ressigning more (less) resources to the subdivision with higher (lower) prices. This type of decomposition has been discussed by, for example, Kornai and Liptak [10] and Kate [19]. This approach is also applied to the block angular system as Dantzig and Wolfe did, and can be viewed as a variant of Benders' decomposition method [13].

Many later decomposition works can be viewed as extensions or modifications of these two basic approaches geared to complex, realistic situations or to make use of inherent structural characteristics ([8], [13], [16]).

In parallel with the researches on the decomposition theory as a mathematical tool, a related research community has explored the applicability and implementability of these mathematical framework to the decentralized decision making of multidivisional organizations (see [2], [3], [4], [8], [18], for example). It turns out that these basic schemes are not well compatible with the decentralized decision making environments in two ways : first, the information flow schemes embedded by the above approaches are not flexible enough to be compatible with the information flow patterns already existing in the systems, and the second, which is particulary notable in the linear porogramming framework, is the discrepancy between the subdivision's optimal activity patterns under the centralized planning (not being decomposed) and those under the decentralized fashion with the optimal price information given. This second phenomena can be easily confirmed from the fact that the optimal activity pattern under the decomposition is always attained at some extreme points or rays of the subdivision's own constraint set, while that under the central planning is normally materialized at non-extremal points due to the existence of linking constraints.

The mixed (hybrid) approach is a partial answer to this. The information flow can be flexible here. That is, the center can choose to provide price information to some subdivisions and quantity information to others. It can also choose to control some resources via prices and others via quantities. Since the mixed decomposition methods require to generate both the prices and the quantities, many works set up two separate master problems: one (primal master) to generate the prices and the other (dual master) to generate the quantities (for example, symmetric nonlinear program of Kronsjö [11], doubly linked *LP* of Stahl [17], and biangular structured *LP* of Eto [6]. A few works attempted to use the single master, instead. For a simple block angular structured *LP* with combining resource constraints, Obel [13] and Burton and Obel [3, 4] utilized a single 'hybrid' master problem rather than two master problems. To be more realistic, we also set up a single master problem which generates prices and quantities simultaneously.

To our knowledge, all previous mixed approaches takes the from of 'vertical' decomposition or 'horizontal decomposition', but not both. That is, the center provides to a specific subdivision either all prices or all quantities (the vertical decomposition), or for a given linking resource the center provides either the price to all subdivisions or the quantity to all subdivisions (the horizontal decomposition).

The mixed decomposition approach of this paper allows the simultaneous application of 'vertical' and 'horizontal' decompositions, that is, the center can choose to provide to sudbivision 1 the price for linking resource A and the quantity for resource B, while it provides to subdivision 2 the quantity for resource A and the price for resource B. In this way, our approach can be set up so that it is compatible with information flow pattern already existing in the organization.

To best illustrate our approach, we have taken a particular type of the block angular type *LP* problem. Consider a network of process models (sudbivisions) each of which is a *LP* type representation. Each node can be represents the input/output relation between subdivisions. Thus, in this network the output of each division is used as inputs to other subdivisions. This type of system is typical in many interorganizational settings, multidivision firms, inter-industry analysis and world trade models.

To formulate this system, we introduce some notations. First, suppose there are N subdivisions in the system. Let x_j denote the activity level vector and c_j be the corresponding cost vector for subdivision j. Denote also the constraints local to subdivision j as $T_jx_j=t_j$. Denote also the price vector of outputs from subdivision j as p_j, and the initial inventory vector of these outputs available in the system as b_j. The matrix A_{ij} represents requirements of the outputs from i needed for the unit operation (x_j) of subdivision j (note that A_{jj} represents the negative of the unit operation output level vector from subdivision j), and the vector b_{ij} indicates the amounts of subdivision i's outputs delivered to subdivision j (in particular, the vector b_{jj} denotes the negative of subdivision j's output vector). Assume each matix and vector have appropriate dimensions. It is noted that all subscripted notations above are either matrices or vectors, not scalars.

Then, the systemwide cost minimzing production pattern of this network of process models can be summarized as the following block angular *LP* problem.

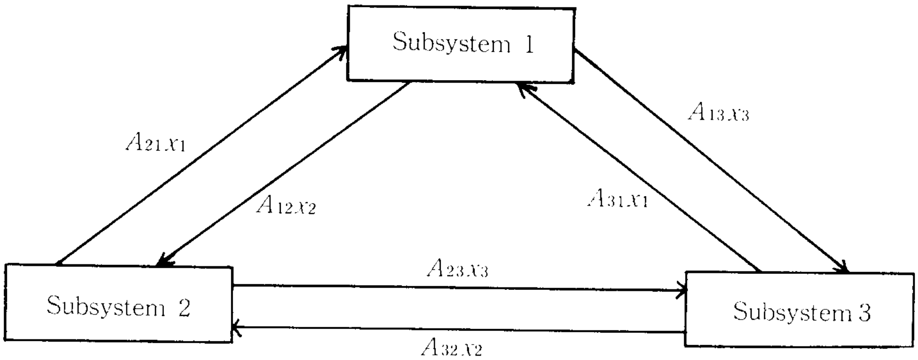

Figure 1. Simple Input/Output Relationship among Subsystems

(*P*) minimize $\sum_{j=1}^{N} c_j x_j$

subject to $\sum_{j=1}^{N} A_{ij} x_j \leqq b_i$ for $i=1,\ldots, N$, (p_i : dual vectors)

$T_j x_j = t_j$ for $j=1,\ldots, N$, (τ_j : dual vectors)

and $x_j \geqq 0$ for $j=1,\ldots, N$.

Then, each sudbivision of this problem can take one of the following forms depending upon the information flow pattern with the center :

Price-quided Subproblem (Dantzig and Wolfe type) : subdivision j receives from the center only the price information for both the outputs and all the inputs (the outputs of other divisions)

(*PS*) Minimize $(c_j + \sum_{i=1}^{N} p_i A_{ij}) x_j$

subject to $T_j x_j = t_j$ (τ_j)

and $x_j \geqq 0$

Resource-quided Subproblem (Ten Kate type) : subdivision j receives from the center the output quota (b_{jj}) and the allowed input quota (b_{ij}, $i \neq j$)

(*RS*) Minimize $c_j x_j$

subject to $A_{ij} x_j \leqq b_{ij}$ for $i=1,\ldots, N$, (p_i)

$T_j x_j = t_j$ (τ_j)

and $x_j \geqq 0$

Other than these two conventional subproblem formulations, we consider the following two alternatives.

Input-price-quided Output-resource-quided Subproblem : subdivision j receives the output b_{jj} and the input prices p_i's from the center

$$(PRS) \quad \text{Minimize} \quad (c_j + \sum_{i \neq j} p_i A_{ij}) x_j$$

$$\text{subject to} \quad A_{jj} x_j \leq b_{jj} \qquad (p_j)$$

$$T_j x_j = t_j \qquad (\tau_j)$$

$$\text{and} \quad x_j \geq 0$$

Input-resource-quided Output-price-quided Subproblem : subdivision j receives the input allowances b_{ij}'s $(i \neq j)$ and the output price p_j from the center

$$(RPS) \quad \text{Minimize} \quad (c_j + p_j A_{jj}) x_j$$

$$\text{subject to} \quad A_{ij} x_j \leq b_{ij} \text{ for } i \neq j \qquad (p_i)$$

$$T_j x_j = t_j \qquad (\tau_j)$$

$$\text{and} \quad x_j \geq 0$$

Most mixed decomposition approaches do use the pure subproblem formulations (PS) and (RS), rather than (PRS) or (RPS). Our approach utilize (PRS), believing it is more commonly found in actual applications (even though the symmetric treatment is possible with (RPS)). Since (PRS) requires both the prices and the quantities, the master(s) should generate both. We present two cases : one with two master problems (one producing the price information the other generating the quantity information) and the other with one hybrid master problem.

In the next section, the two master problems for (P), primal and dual, will be developed, and the resulting decomposition method and its convergence will be discussed. Section 3 presents another method which requires only one master problem generating both prices and quantities. Convergence discussion utilizes the results of Section 2. An illustrative example and computational issues will be covered in Section 4.

2. Hybrid Decomposition Method Using Two Master Problems

The hybrid decomposition method to be discussed here applies price control to the inputs and quantity control to the outputs of each subsystem. That is, we use the subproblem formulation (PRS) rather than (RPS). We will use the similar notation as before in the following representation. But the notations for prices and quantities are further differentiated by attaching the upper "$^-$" or "$^\sim$" as is needed, where the former denotes the information reported by the subproblems and the latter is that given from the masters, respectively.

Before introducing our hybrid master problem, we consider the following decomposition scheme to coordinate the (PRS). In what follows, two master problems are derived for investigating the coordination process. The first master problem is Dantzig-Wolfe type of (P) :

$$(PM) \quad \text{Minimize} \quad \tilde{\phi} = \sum_{j=1}^{N} \sum_{r \in R_j} c_j \bar{x}_j^r \lambda_{jr}$$

$$\text{subject to} \quad \sum_{j=1}^{N} \sum_{r \in R_j} A_{ij} \bar{x}_j^r \lambda_{jr} \leq b_i \text{ for } i = 1, \ldots, N, \qquad (p_i)$$

$$\sum_{r \in R_j} \lambda_{jr} = 1 \text{ for } j = 1, \ldots, N, \qquad (z_j)$$

and $\lambda_{jr} \geqq 0$ for $j = 1, \ldots, N$ and $r \in R_j$,

where R_j is the index set of the accumulated proposals x_j's submitted from the subsystem j.

The other master problem is dual one, which can be derived from the dual of (P). It can be written as follows:

(DM) Maximize $\tilde{\psi} = \sum_{j=1}^{N} \sum_{s \in S_j} (\bar{\tau}_j^s t_j - \bar{p}_j^s b_j) \mu_{js}$

subject to
$$\sum_{s \in S_j} \bar{\tau}_j^s T_j \mu_{js} - \sum_{i=1}^{N} \sum_{s \in S_i} \bar{p}_i^s A_{ij} \mu_{is} \leqq c_j \text{ for } j = 1, \ldots, N, \qquad (x_j)$$

$$\sum_{s \in S_j} \mu_{js} = 1 \text{ for } j = 1, \ldots, N, \qquad (w_j)$$

and $\mu_{js} \geqq 0$ for $j = 1, \ldots, N$, and $s \in S_j$,

where S_j is the index set of the accumulated proposals p_j's and τ_j's submitted from the subsystem j.

Now we can develop a decomposition procedure to obtain an optimal solution to (P) using subproblem formulation (PRS). It can be summarized as follows.

Algorithm

Step 0 (Initialization)

Initially guess $(\bar{b}_{ij}^0, \bar{p}_i^0, i \neq j)$ for each j and solve (PRS). Let $(\bar{x}_j^0, \bar{p}_j^0)$ denote the optimal primal dual solutions from each j and set R_j^1 and $S_j^1 = \{0\}$ for every j and $k = 1$.

Step 1 (Iteration k)

If $R_j^k = R_j^{k-1}$ and $S_j^k = S_j^{k-1}$ for all j, stop.

Otherwise solve (PM) and (DM), let $(\bar{p}_j^k, \bar{q}_j^k)$, $j = 1, \ldots, N$ denote the optimal dual solutions $(\bar{q}_j^k = b_i - \sum_{i \neq j} A_{ji} \bar{x}_i^k)$. If $\tilde{\phi}^k = \tilde{\psi}^k$ stop, otherwise go to Step 2.

Step 2

For each $j = 1, \ldots, N$,

If $(\bar{q}_j^k, \bar{p}_i^k) = (\bar{q}_j^{k-1}, \bar{p}_i^{k-1})$ for each $i \neq j$, then next j.

Otherwise solve (PRS) with $\bar{b}_{ij} = \bar{q}_j^k$ and $\bar{p}_i = \bar{p}_i^k$, for each $i \neq j$.

Let $(\bar{x}_j^k, \bar{p}_j^k)$ denote the optimal solution of the j-th subsystem. If there does not exist $r \in R_j$ such that $\bar{x}_j^r = \bar{x}_j^k$ then let $R_j^{k+1} = R_j^k \cup \{k\}$, and similarly update $S_j^{k+1} = S_j^k \cup \{k\}$. Repeat to next j.

Set $k = k + 1$ and return to Step 1.

The above algorithm can solve the original problem (P). In this case, however, the feasible regions of (PRS) and its dual vary in successive iterations. This fact implies that the general finite convergence of pure LP decomposition procedures (which generates proposals only from the extreme point of subproblem feasible sets) cannot be applied any more. This seems to be a drawback from the computational standpoint, but this is the trade-off we have to accept since our approach, for that very reason of infiniteness, provides the decentralized optimal decisions with are compatible with the systemwide decision (recall this is not available in Dantzig-Wolfe method).

Now we show the convergence of the proposed method.

Lemma 1. At any iteration step, the optimal objective value of (PM) and (DM) satisfy the follwing relation:

objective of $DM(\tilde{\psi}) \leqq$ optimal objective of $(P) \leqq$ objective of $PM(\tilde{\phi})$.

Proof. It is helpful to consider the restrictive nature of the both masters. The feasible activity set of (PM) is the subset of that of (P). Similar inner linearization arguments can be applied to (DM) and the dual of (P), so as to confirm the left inequality. ///

Lemma 2. In the successive iteration steps k and $(k+1)$, the following relations will hold.

1) $\tilde{\phi}^k = \sum_j \tilde{z}_j^k - \sum_j \tilde{p}_j^k b_j \leqq \sum_j (c_j + \sum_i \tilde{p}_i^k A_{ij}) \bar{x}_j^k - \sum_j \tilde{p}_j^k b_j$

2) $\tilde{\psi}^k = \sum_j (c_j \tilde{x}_j^k + \tilde{w}_j^k)$

$\geqq \sum_j (c_j - \bar{\tau}_j^k T_j + \sum_i \bar{p}_i^k A_{ij}) \bar{x}_j^k + \sum_j \bar{\tau}_j^k t_j - \sum_j \bar{p}_j^k b_j$

3) $c_j \bar{x}_j^{k+1} + \sum_{i \neq j} \tilde{p}_i^k A_{ij} \bar{x}_j^{k+1} = -\bar{p}_j^{k+1} \tilde{b}_{jj}^k + \bar{\tau}_j^{k+1} t_j$

$= \sum_{i \neq j} \bar{p}_j^{k+1} A_{ji} \bar{x}_i^k + \bar{\tau}_j^{k+1} t_j - \bar{p}_j^{k+1} b_j$

4) $c_j \geqq \bar{\tau}_j^{k+1} T_j - \bar{p}_j^{k+1} A_{jj} - \sum_{i \neq j} \tilde{p}_i^k A_{ij}$

Proof. Note that the iteration indices k and $k+1$ above imply that the subproblems are solved first. Thus 1) and 2) can be derived from the strong duality and dual feasibility conditions of the two master problems. And then 3) denotes the primal/dual objective values of *j-th* subproblem while 4) means the dual feasibility. ///

Theorem 1. If (P) has a finite optimal solution, the values obtained $\tilde{\phi}$ and $\tilde{\psi}$ converge to the optimal objective value of (P).

Proof. Since it is assumed that (P) and its dual are feasible, (PM) and (DM) can be feasible with some proposals. As iteration proceeds (R_j and S_j grow), $\tilde{\phi}$ decreases and $\tilde{\psi}$ increases monotonically. By their monotonicity and boundedness (from the above Lemma 1), $\lim \tilde{\phi}$ and $\lim \tilde{\psi}$ exist. Now it is sufficient to show that $\lim \tilde{\phi} = \lim \tilde{\psi}$.

First, let $\tilde{\phi}^*$ and $\tilde{\psi}^*$ be the limit points of the sequences $\{\tilde{\phi}^k\}$ and $\{\tilde{\psi}^k\}$. Now suppose that $\tilde{\phi}^* > \tilde{\psi}^*$ and take $\varepsilon > 0$ such that $\varepsilon < (\tilde{\phi}^* - \tilde{\psi}^*)/2$. For iteration k, let $D_j = \tilde{z}_j^k - (c_j + \sum_i \tilde{p}_i^k A_{ij}) \bar{x}_j^{k+1}$, which is the negative of the reduced cost for subdivision j's proposal to the (PM).

After the optimality test of (PM) at iteration $(k+1)$, D_j may be positive for some j. This amount is an upper bound of initial improvement of master objective value (since the proposal's weight $\lambda_{j(k+1)}$ is less than or equal to 1), but we can have $\sum_j D_j < \varepsilon$ for sufficiently large k

(See **Appendix** to confirm the reason in detail). From this fact and 1) of Lemma 2, we can write

$$\bar{\phi}^k - \varepsilon < \sum_j (c_j + \sum_i \bar{p}_i^k A_{ij}) \bar{x}_j^{k+1} - \sum_j \bar{p}_j^k b_j.$$

Similar arguments can be applied to (*DM*) and we can get the relation,

$$\tilde{\psi}^k + \varepsilon > \sum_j (c_j - \bar{\tau}_j^{k+1} T_j + \sum_i \bar{p}_i^{k+1} A_{ij}) \tilde{x}_j^k + \sum_j \bar{\tau}_j^{k+1} t_j - \sum_j \bar{p}_j^{k+1} b_j.$$

Merging above two inequlities by the assumption on ε and substitute some arguments by 3) and 4) of Lemma 2:

$$\begin{aligned} 0 < \tilde{\phi}^k - \tilde{\psi}^k - 2\varepsilon &< \sum_j \tilde{p}_j^k A_{jj} \bar{x}_j^{k+1} - \sum_j \tilde{p}_j^k b_j - \sum_j (c_j - \bar{\tau}_j^{k+1} T_j + \bar{p}_j^{k+1} A_{jj}) \tilde{x}_j^k \\ &\leqq \sum_j \tilde{p}_j^k A_{jj} \bar{x}_j^{k+1} - \sum_j \tilde{p}_j^k b_j - \sum_j \sum_{i \neq j} \tilde{p}_i^k A_{ij} \tilde{x}_j^k \\ &= \sum_j \tilde{p}_j^k (b_j - \sum_{i \neq j} A_{ji} \tilde{x}_i^k) - \sum_j \tilde{p}_j^k b_j + \sum_j \sum_{i \neq j} \tilde{p}_i^k A_{ij} \tilde{x}_j^k \\ &= 0. \end{aligned}$$

This contradiction completes the proof. ///

3. Hybrid Decomposition with a Single Master Problem

Now we can introduce our hybrid major decomposition scheme with the preliminary works above. In order to relate (*DM*) to (*PM*), let (*DDM*) be the dual of (*DM*) as follows:

(*DDM*) Minimize $\psi = \sum_{j=1}^{N} c_j x_j + \sum_{j=1}^{N} w_j$

subject to

$$w_j + \bar{\tau}_j^s T_j x_j - \sum_{i=1}^{N} \bar{p}_j^s A_{ji} x_i \geqq \bar{\tau}_j^s t_j - \bar{p}_j^s b_j \text{ for } s \in S_j \text{ and } j = 1, \ldots, N \qquad (\mu_{js})$$

and $x_j \geqq 0$, w_j unrestricted for $j = 1, \ldots, N$

Note that (*PM*) and (*DDM*) are of similar form, where the former takes inner linearization-restriction and the latter takes outer linearization-relaxation according to Geoffrion [7]'s criterion. Now we merge two master problems into one hybrid master problem (*HM*) as follows:

(*HM*) Minimize $\tilde{\pi} = \sum_{j=1}^{N} \sum_{r \in K_j} c_j \bar{x}_j^r \lambda_{jr} + \sum_{j=1}^{N} w_j$

subject to

$$\sum_{i \neq j} \sum_{r \in K_i} A_{ji} \bar{x}_i^r \lambda_{ir} + q_j \leqq b_j \qquad \text{for } j = 1, \ldots, N \qquad (p_j)$$

$$w_j + \bar{p}_j^s (\sum_{r \in K_j} A_{jj} \bar{x}_j^r \lambda_{jr} - q_j) \geqq 0 \qquad \text{for } s \in K_j \text{ and } j = 1, \ldots, N \qquad (\mu_{js})$$

$$\sum_{r \in K_j} \lambda_{jr} = 1 \qquad \text{for } j = 1, \ldots, N \qquad (z_j)$$

and $\lambda_{jr} \geqq 0$ for $r \in K_j$, w_j unrestriced, $q_j \geqq 0$ $j = 1, \ldots, N$.

where K_j is the index set of proposals fom subsystem j.

The manipulation strategy to obtain the (HM) is applying the approximation $x_j = \sum_{r \in K_j} \bar{x}_j^r \lambda_{jr}$ for each j into (DDM). Then since $T_j \bar{x}_j^r = t_j$ holds for all j, the (μ_{js}) rows become simpler than that of (DDM). And the (μ_{js}) rows are restricted via (p_j) rows and q_j quantity vectors. On the other hand, (HM) is a relaxed version of (PM) on (p_j) rows. Below we describe the hybrid decomposition method.

Algorithm

Step 0 (Initialization)

Initially quess $(\bar{b}_{jj}^o, \bar{p}_i^o, i \neq j)$ for each j and solve (PRS). Let $(\bar{x}_j^o, \bar{p}_j^o)$ denote the optimal primal dual solutions from each j and set $K_{j1} = \{0\}$ for every j and $k = 1$.

Step 1 (Iteration k)

If $K_j^k = K_j^{k-1}$ for all j, stop.

Otherwise solve (HM) and let $(\tilde{q}_j^k, \tilde{p}_j^k)$, $j = 1, \ldots, N$ denote the optimal primal dual solutions and go to Step 2.

Step 2

For each $j = 1, \ldots, N$,

If $(\tilde{q}_j^k, \tilde{p}_i^k) = (\tilde{q}_j^{k-1}, \tilde{p}_i^{k-1})$ for each $i \neq j$, then next j.

Otherwise solve (PRS) with $\bar{b}_{jj} = \tilde{q}_j^k$ and $\bar{p}_i = \tilde{p}_i^k$, for each $i \neq j$.

Let $(\bar{x}_j^k, \bar{p}_j^k)$ denote the optimal solution of the j-th subsystem. If there exists $h \in K_j$ such that $(\bar{x}_j^h, \bar{p}_j^h) = (\bar{x}_j^k, \bar{p}_j^k)$, then next j,

otherwise let $K_j^{k+1} = K_j^k \cup \{k\}$, next j.

Set $k = k + 1$ and return to Step 1.

In the above algorithm, we ignored the possibility of case that (PRS) is infeasible (dual unbounded solution). But the simple treatment of this possibility (generating new cut using phase I multipliers) can be applied as in [13] and [19]. The algorithm utilizes the primal and dual information simultaneously and find an optimal solution to (P). Now we can investigate the convergence property of the proposed algorithm using the Lemmas and Theorem given before.

Theorem 2. At any iteration, if the given information from subsystems for the two master method (to (PM) and (DM))and for the single master method (to (HM)) are same (R_j and S_j are equivalent to for K_j for all j), then the following inequality holds:

$$\tilde{\psi} \leqq \tilde{\pi} \leqq \tilde{\phi}$$

Further, at least one of the two inequalities is strict unless the final optimality is obtained.

Proof. First we show left inequality. Since the feasible region of (DDM) contains that of (HM) and the objective is to minimize, it trivially holds. Next, the right inequality holds as follows. Let $\tilde{\lambda}_{jr}$ be optimal to (PM) and substitute it into (HM) with $q_j = \sum_{r \in K_j} A_{jj} \bar{x}_j^r \tilde{\lambda}_{jr}$.

Then since $\tilde{w}_j \geqq \bar{p}_j^s \ (b_j - \sum_i \sum_{r \in K_i} A_{ji} \bar{x}_i^r \tilde{\lambda}_{ir})$, $\bar{p}_j^s \geqq 0$, and the term in the parenthesis is nonpositive, $\tilde{w}_j \leqq 0$ can always be attained for all j at the optimum of (HM). The desired result is obtained.

Finally, since $\tilde{\psi} < \tilde{\phi}$ before the optimal solution of (P) is reached, at least one of the above two inequalities must hold strictly. This completes the proof. ///

Theorem 3. If the original problem (P) has a finite optimum solution, the solutions generated by the hybrid master problem converge to the optimum of (P).

Proof. Consider primal and dual solution of (HM), which can be used for new quantity and price for the (PRS). And the (PRS) can generate new proposals to (PM) and (DM), although they are not proposed under the control of (PM) and (DM). Here we can say that the proposals under the control of (HM) do not interrupt the monotonicity of solutions (PM) and (DM) since the two masters can disregard these proposals if it doesn't help to improve respective objective value. Now we will show that the sequences $\{\tilde{\phi}^k\}$ and $\{\tilde{\psi}^k\}$ generated under (HM)'s control also converge to $\tilde{\phi}^* = \tilde{\psi}^*$. As discussed in proof of **Theorem 1** and **Appendix,** for sufficiently small $\varepsilon < 0$ and sufficiently large iteration counter k, if $\tilde{\phi}^k - \tilde{\phi}^{k+1} \ll \varepsilon$ and $\tilde{\psi}^{k+1} - \tilde{\psi}^k \ll \varepsilon$, then the subproblem's proposal must be located in some previous proposal's ε-neighborhood.

So we can derive $|\ \tilde{\pi}^k - \tilde{\pi}^{k+1}\ | < \varepsilon$. Now consider the solutions $(\tilde{p}^k, \tilde{z}^k)$ of (DDM) and $(\hat{p}^k, \hat{z}^k)$ from the dual of (HM). If we substitute $(\hat{z}^k, \hat{p}^k)$ into D_j defined in the proof of Theorem 1 and redenote D_j as $\hat{D}_j$, the new quantity $\sum_j \hat{D}_j < \varepsilon$ is established. The remaining arguments can be applied in the similar manner as in Theorem 1. Hence the conclusion $\tilde{\phi}^* = \tilde{\psi}^* = \tilde{\pi}^*$ can be obtained. ///

Remarks

1. Since (HM) generates objective values closer to that of (P) than (PM) and (DM) at every iteration step, it is expected that the hybrid decomposition procedure converges more rapidly.
2. Another advantage of the proposed method over the two master case is that it solves only one master problem rather than two. This single master problem do utilize almost twice as much information as pure decomposition procedures do.
3. Many authors adressed that main drawback of conventional decomposition approaches in light of decentralized decision process is the lack of "autonomy" (e.g., [6], [8]), that is, a subproblem's optimal activity pattern given an optimal price information does not in general guarantee the systemwide material balances. In our hybrid approach, the autonomy of subsystems in terms of compatibility of local decision with those of global optimal primal/dual solutions can be accomplished. This is possible because the primal and dual guidelines given to subsystems are consistent in (HM) and the proposals submitted will satisfy the primal/dual constraints of (HM) as well, and hence those of (P).
4. Economic interpretation of the control mechanism of the master problem with respect to the price and quantity coordination: In the course of iterations, the master problem can generate solutions with either $w_j = 0$, $w_j > 0$, or $w_j < 0$. If $w_j = 0$, the flow feasibility is obtained, so that next proposal from that subsystem is to be used for reduction of processing cost only. If $w_j > 0$, there exists shortage in the outputs of subsystem j. Then the master of the next iteration requests subsector j to produce more quantity and guides other

subsectors to reduce their consumption of the output of j via the higher price. Similar coordination of reverse direction also can be exercised and global feasible and optimal solution is attained eventually.

4. Illustrative Example

We illustrate the proposed approach of Section 3 via the following simple example. Consider the system with subdivision 1 and subdivision 2, trading their products each other. Suppose there are no initial inventories of these products. x_1 and x_2 are activity levels of subdivision 1, and y_1 and y_2 are those of subdivision 2. The first row represents the material balance for the output of subdivision 1 (produced by subdivision 1 and consumed by subdivision 2), while the second shows that for the subdivision 2's output. These two rows are linking resource constraints.

$$\begin{array}{lrl}
\text{Minimize} & 3x_1 + 5x_2 + 5y_1 + 3y_2 & \\
\text{subject to} & 4x_1 + 5x_2 - 5y_1 - 3.5y_2 & \geqq 0 \\
 & -4x_1 - 4x_2 + 4y_1 + 5y_2 & \geqq 0 \\
 & 3x_1 + 4x_2 & \leqq 150 \\
 & 4x_1 + 6x_2 & \geqq 100 \\
 & 2.5y_1 + 5y_2 & \leqq 150 \\
 & 6y_1 + 3y_2 & \geqq 100 \\
\text{and} & x_1, x_2, y_1, y_2 \geqq 0 &
\end{array}$$

Note that this problem is a block angular type *LP.* Now we solve this by the hybrid decomposition method of Section 3 utilizing the subproblem formulation (*PRS*).

$$\begin{array}{lrl}
\text{Minimize} & (3 + 4p_2)x_1 + (5 + 4p_2)x_2 & \\
\text{subject to} & 4x_1 + 5x_2 & \geqq q_1 \\
 & 3x_1 + 4x_2 & \leqq 150 \\
 & 4x_1 + 6x_2 & \geqq 100 \\
\text{and} & x_1, x_2 \geqq 0 &
\end{array}$$

$$\begin{array}{lrl}
\text{Minimize} & (5 + 5p_1)y_1 + (3 + 3.5p_1)y_2 & \\
\text{subject to} & 4y_1 + 5y_2 & \geqq q_2 \\
 & 2.5y_1 + 5y_2 & \leqq 150 \\
 & 6y_1 + 3y_2 & \geqq 100 \\
\text{and} & y_1, y_2 \geqq 0 &
\end{array}$$

The results of the iterations are summarized in Table 1, where an optimal solution is found at iteration 7. We also presented optimal solutions obtained without decomposition as well in Table 2. It should be noted that the optimal activity levels obtained without decomposition is not necessarily same as those with decomposition, even though the system's total optimal costs are same. It is also noted that the former solution is not necessarily compatible with the subdivision's decentralized optimal production pattern, while the latter confirms to the subdivision's optimal behavior. These could be important implications in applying to decentralized decision environments with existing control information patterns. This implementational feature is not shared in the Dantzig-Wolfe's method.

Note also in Table 1 that prices informed from the master and those reported by the subdivisions are equal at termination, and that the quantities from the master and from the subdivisions are equal provided the associated prices are positive.

Table 1. Summary of Iterations

Iteration	Master Obj.	$\bar{p}_1$	$\bar{p}_2$	$\bar{q}_1$	$\bar{q}_2$	$\tilde{p}_1$	$\tilde{p}_2$	$\tilde{q}_1$	$\tilde{q}_2$
0						0.000	0.000	120.00	120.00
1	168.056	0.750	0.167	101.11	120.00	0.750	0.167	101.11	120.00
2	164.907	0.917	0.417	101.11	101.11	0.917	0.167	101.11	101.11
3	160.185	0.917	0.472	94.81	101.11	0.750	0.417	94.81	101.11
4	162.809	0.333	0.417	101.11	89.63	0.333	0.278	94.81	101.11
5	164.558	0.056	0.278	94.81	89.63	0.056	0.184	94.81	101.11
6	163.889	0.000	0.185	101.11	100.00	0.000	0.167	100.00	100.00
7	163.889	0.000	0.167	94.44	100.00				

Table 2. Summary of Final Solutions

	Objective	Cost	x_1	x_2	y_1	y_2	q_1	q_2	p_1	p_2
(P)	163.889	163.889	25.0	0.0	11.1	11.1	94.4	100.	0.0	0.167
(HM)	163.889	166.667*	25.0	0.0	8.3*	16.7*	100.	100.	0.0	0.167
(PRS)	163.889	163.889	25.0	0.0	11.1	11.1	94.4	100.	0.917*	0.167

*'s denote that the alternative optimal solution same as in (P) can be obtained.

** In (HM), $w_2 = -2.77778$ is the source of the difference between obj. value and total cost.

5. Conclusion

In this paper, we have considered ways of decomposing and integrating a set of activity analysis models each interacting with others via input/output relations. Since this problem is block angular type, conventional decomposition methods can well be applied to find optimal solutions. We have suggested a new hybrid decomposition scheme where one hybrid master problem jointly deals primal and dual information from subsystem. We have shown the proposed method generates the better suboptimal solutions (guranteeing decentralized autonomy) than the pure decompositions.

Main motivation of this research was to provide a flexible yet implmentable decomposition method that has been needed in the efforts to utilize the mathematical programming's decomposition theory in the decentralized organizational decision making. One that follows this work would be the decentralized decision making without master. In other words, if each subdivision is cooperative in the sense that its objective is compatible with the systemwide goal, you could reach the overall optimum even without the center's coordination. This can also be interpreted as that each subdivison play the role of the center in turn. This type of decomposition theory can be applied even in the organization's management style in addition to the decentralized decision issues. This work is under investigation and will appear in a subsequent paper.

Appendix

We must show that we can freely choose sufficiciently large k such that $D_j = \tilde{z}_j^k - (c_j + \sum_i \tilde{p}_i^k A_{ij})\bar{x}_j^{k+1} < \delta$ for any given $\delta > 0$. Thus we confirm $\sum_j D_j < \varepsilon$ for $\varepsilon < 0$ with $\delta = \varepsilon / N$.

Proof. First consider the feasible set of (PRS), which has only finitely many rows. Thus after sufficiently large number of iterations are completed, some proposals from (PRS) are BFS's defined by the same set of binding rows only different in their $\tilde{b}_{jj}$ constants. In other words, if $\bar{x}_j^{k+1}$ and $\bar{x}_j^{r+1}$ $(r < k)$ are using the same basis $\bar{B}_j$, then they can be denoted $\bar{B}_j[\tilde{b}_{jj}^k, t_j]$ and $\bar{B}_j[\tilde{b}_{jj}^r, t_j]$, respectively. But if the iteration counter k and r are sufficiently large, $(\tilde{\psi}^k - \tilde{\psi}^r) \rightarrow 0+$ as needed. Recall that $\tilde{b}_{jj}$'s are computed from the dual solution $\tilde{x}$ of (DM). Even if (DDM) has alternative bases with the same objective value $\tilde{\psi}^*$, their number is finite since the number of (DM)'s row is finite. Hence we can choose k and r such that $\bar{x}_j^{k+1}$ and $\bar{x}_j^{r+1}$ are using the same basis and furhter $\tilde{x}^k$ and $\tilde{x}^r$ converge to the same limiting point. This observation confirms us that the distance between $\bar{x}_j^{k+1}$ and $\bar{x}_j^{r+1}$ converges to zero as needed. But we have $D_j' = \tilde{z}_j^k - (c_j + \sum_i \tilde{p}_i^k A_{ij})\bar{x}_j^{r+1} \leqq 0$ since $r < k$. Comparing D_j and D_j', the desired result follows. ///

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