170 Transaction of the Korean Society of Mechanical Engineers, Vol. 10, No. 1, pp. 170~180, 1986.

<Original>

On the Computer Simulation for the Third Integral and an Application of the Poincaré Map in Hamiltonian System†

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(Received July 24, 1985)

Hamiltonian 비선형계의 기하학적 연구와 제 3 의 운동상수 응용

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Key Words : Normal Mode Vibration(정규모우드진동), Bifurcate(분기), Integral of Motion (운동상수), Poincaré Map(Poincaré 사상), Perturbation Technique(섭동기법)

초 록

2 자유도 Hamiltonian 운동계에서 비선형 정규모우드(normal mode)들의 안정성을 예측하기 위한 제 3 의 운동상수를 선형계의 진동수비가 1 : 1 이고 포텐셜이 4 차항까지 우함수인 일반계에 적용하여 발전시켰다. 이는 Hamiltonian 을 정규모우드로 바꾸는 B-G 변환과 함수들을 부호처리함과 Poincaré map 을 이용하다.

비선형계에서 비선형상수에 따라 모우드가 bifurcate 되며, 각각의 모우드 안정성은 제 3 의 운동상수와 Poincaré map 으로 정확히 판정할 수 있다는 결론을 얻었다.

1. Introduction

The qualitative analysis of differential equations have beentworked by Poincaré (1880, 1890 1899), Birkhoff[5], Liapunov (1949), Arnold[3,4]. Smale outlined a number of outstanding problems and stimulated much of his work[16]. In general, the integral of a time-independent Hamiltonian system about an equilibrium point do not exist (Poincaré, 1957). However, Henon and Heiles[9] represented a regular behavior of the flow, indicating the existence of approximate integrals. The flow is the vector fields (or phase space).

Later, F.G. Gustavson[8] constructed formal integrals of a Hamiitonian system near an equilibrium points. He proved that there exists

† Presented at KSME July Conference 1984.

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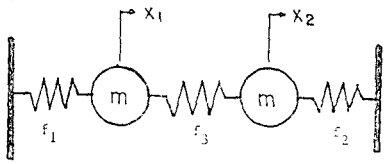
a formal canonical transformation that reduce the Hamiltonian to a normal form which is an integral of a motion. Particularly, he treated two-degree-of-freedom Hamiltonian system that is plotted in level lines of the intersection of the integral with some appropriate surface. These expressions are based on KAM(Kolmogorov-Arnold-Moser) theory that shows the existence of invariant surfaces for nearly integr-a ble Hamiltonian system.

In addition to these facts, the qualitative picture of the flows is called Poincaré map[7,10,11]. The map is utilized to predict the stability of NNM's in autonomnus two-degree-of-freedom Hamiltonian sysem. The procedure is based on using the Birkhoff-Gustavson(B-G) transformation to obtain an approximation for the Poincaré map[11].

We summarize previous results concerning the concept of NNM's. Rosenberg[15] introduced the concept of normal modes of a nonlinear system(Appendix). The existence of NNM's is given by C.H. Pak(1968). Rosenberg[15] and Anand[2] noted that for a class of duall-mass system when the nonlinear spring forces are of homogeneous degrees in the deflections there exist more two normal modes. Later it is shown (18) that the total number of NNM's of a nonlinear duall-mass system must be even number which can only decrease or increase in pairs.

Nontheless, nonlinear systems remained largely inaccessible, a part from successible applications of perturbation methods to weakly nonlinear problems[14,17]. But the celestial mechanics is difficult in applications of perturbation methods[10]. The analysis remained the favored tool for the study of dynamical problems until Poincaré work in 1880. He showed that perturbation methods might not yield correct results in some cases, because the series used in such calculations diverged. Poincaré has studied geometry in the development of a qualitative approach to the study differential equations. Lately, there is a discussion of chaos and non integrability in Hamiltonian systems by Abraham & Marsden[1] and Holmes[7,10]

In this papsr, the computer program is constructed to find the third integrals with higher orders and the integral is applied to two-degree-of-freedom autonomous Hamiltonian system in resonance. We plot the integral on the invariant surface, Poincaré map, and investigate the stability of NNM's in an unsymmetric dual-mass system.



$f_1=d+k_1d^3$, $f_2=d+k_2d^3$, $f_3=k_3d^3$

Fig. 1 The system S

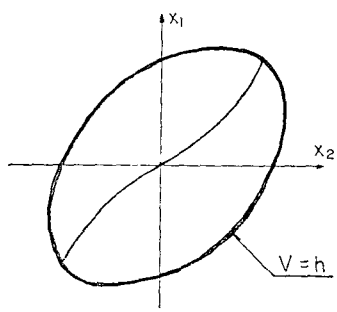


Fig. 2 A nonlinear normal mode

2. The System S

We consider the motion of two unit masses in a nonlinear autonomous two-degree-of-freedom Hamiltonian system S contained to move along a straight line restrained by two different anchor springs and a coupling spring in Fig. 1.

The positions of the masses are given by generalized coordinates x and generalized momentum y, both of which are unstreched. The assumption is that restoring forces f_i for the different anchor springs are given by $f_1=d+$

k_1d^3 and $f_2=d+k_2d^3$, while for the coupling spring, $f_3=k_3d^3$.

The equation of motion for the system S are

$$\dot{x}_1=y_1$$

$$\dot{x}_2=y_2$$

$$\dot{y}_1=\ddot{x}_1=-\frac{\partial V}{\partial x_1}=-x_1-k_1x_1^3-k_3(x_1-x_2)^3,$$

$$\dot{y}_2=\ddot{x}_2=-\frac{\partial V}{\partial x_2}=-x_2-k_2x_2^3-k_3(x_2-x_1)^3,$$

where $V(x_1, x_2)=1/2(x_1^2+x_2^2)+\frac{k_1}{4}x_1^4+\frac{k_2}{4}$

$$x_2^4+\frac{k_3}{4}(x_1-x_2)^4, \qquad (1)$$

The first integral for the system S corresponding to the conservatien of the edergy

$$H(x_1, x_2, y_1, y_2)=\frac{1}{2}\left(y_2^1+y_2^2\right)+V(x_1, x_2)=h, \qquad (2)$$

where $H(x_1, x_2, y_1, y_2)$ is a Hamiltonian.

In a previous work(11, 13) the system S became the identical anchor springs and the coupling spring, $f=f_1=f_2=d+kd^3$, $f_3=d^3$. This system admits similar normal modes (SNM's) if $0\leq k\leq 4$ and 4 SNM's if $k<0$ and $k>4$.

The stability of the $x_2=-x_1$ out-of phase mode and of the two SNM's which bifurcate out of it have been investigated by Mathieu equation[12], and the stability of the $x_2=x_1$ in-phase mode and of the two SNM's bifurcated out of it have been investigated by the third integral with the order four and the Poincaré map[11]. They have written results as follows.

It was given a description of the nature of the invariant curves of the Poincaré map in the neighborhood of the singular points, i.e., NNM's. For the out-of phase mode $x_2=-x_1$ a Taylor series give a family of ellipse if $k<4$ (stable) in Fig. 3, 4 and it is a family of the hyperbolas if $k>4$ (unstable) in Fig. 5. For the in-phase mode $x_2=x_1$ a Taylor series give a family of ellipse if $k>0$ (stable) in Fig. 4, 5 and it is a family of hyperbolas if $k<0$ (unstable) in Fig. 3. Sketches of the Poincaré map

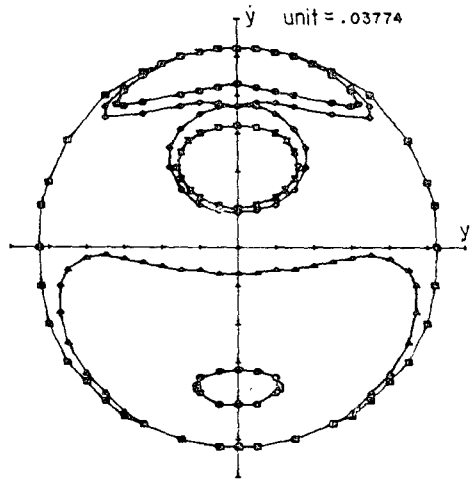


Fig. 3 The third integral $k=-1$, $h=.02$

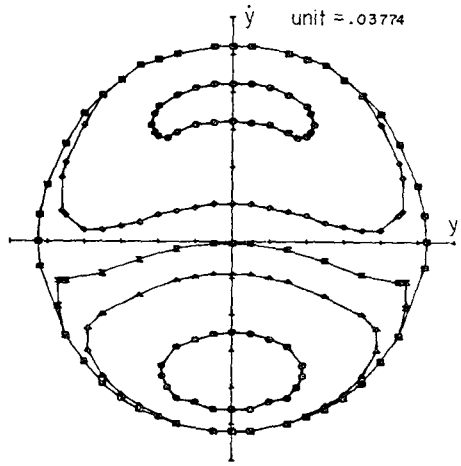


Fig. 4 The third integral $k=1$, $h=.02$

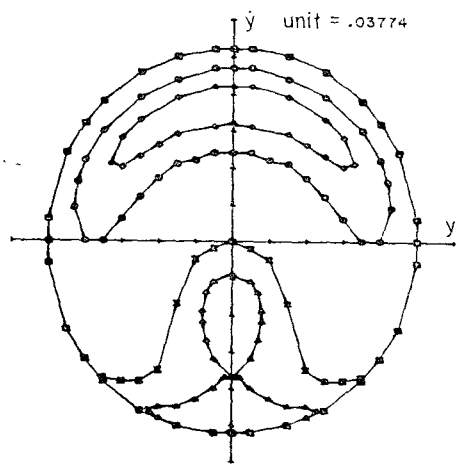


Fig. 5 The third integral $k=5$, $h=.02$

for various values of k are given in Fig. 3, 4, 5[11].

In addition to Month and Rand[11], we construct the third integral with higher orders than four and apply the approximate Poincaré map to the unsymmetric system S.

(1) In the system S with three nonlinear parameters k_1, k_2, and k_3, what is the tendency of NNM's in parametric bifurcations?

(2) What happen to the NNM's at bifurcation point if h is small or large?

(3) Let us plot the Poincaré map by the approximate third integral as calculated by a computer procedure.

3. Hamiltonian Mechanics

In this section it is discussed the Hamiltonan mechanics and Hamilton's canonical equation of motion in terms of generalized coordinates and momenta (q, p). We note that a Hamiltonian mechanical system is given by an even dimensional manifold, a symplectic structure on it and a function on it, and that the Hamiltonian vector fields on a symplectic manifold for a Lie algebra (Poisson bracket). We conclude with a discussion of an invariant tori with respect to the phaseflow.

On a symplectic manifold, there is a natural isomorphism between vector fields and l-forms. Hamiltonian vector field is a vector on a symplectic manifold corresponding to the differential of a function. A vector field on a manifold determines a phase flow, i.e., a one-parameter group of diffeomorphisms. The phase flow of a Hamiltonian vector field on a symplectic manifold preserves the symplectic structure of phase space, in Fig. 6, 7[3,4]

In order to integrate a system of 2n differential equations, there is to be 2n first integrals. However, it is given in (11) that if we are given a canonical system of 2n differential equations it is often sufficient to know only n first integrals -each of them allows us to reduce the order of the system not just by one, but by two.

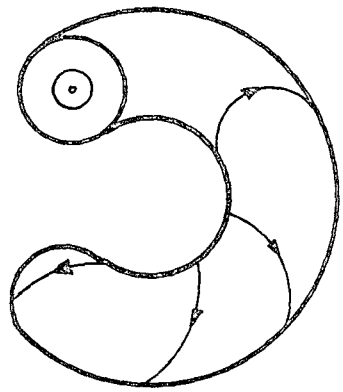
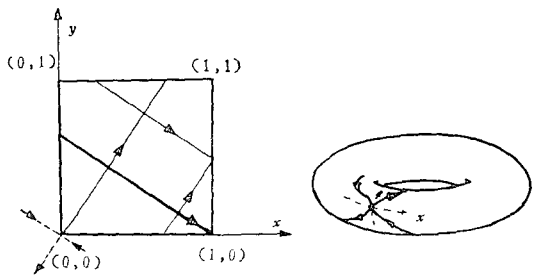


Fig. 6 Invarint tori in a three dimensional energy level manifold



(a) on R^2, the covering space;
(b) on T^2.

Fig. 7 The linear map on the torus

Then we use canonical transformations to develop an approximate first integral independent of the first integral, $H=h$, by independent first integral of the Hamiltonian equations.

We mean $F(q,p)=C$, when F and H are functionally indepenent. If we have a first integral, $F=C$, then

$$\frac{dF}{dt}=0=\frac{\partial F}{\partial q}\frac{dq}{dt}+\frac{\partial F}{\partial p}\frac{dp}{dt}$$
$$=\frac{\partial F}{\partial q}\frac{\partial H}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial H}{\partial q}=\{F,H\}$$

where $\{,\}$ is called the poisson bracket.

We emphasize that the function F is a first integral of a system with the Hamiltonian function H if and only if the poisson bracket

$\{H, F\}=0$

is identically equal to zero.

Now we apply Liouville's theorem on integrable system, i.e., if, in a system with n degrees of freedom (2n dimensional phase space), n independent first integrals in involution are known, then the system is integrable by quadratures.

In two degree of freedom Hamiltonian system if we find an approximate first integral independent of the Hamiltonian we can integrate the system by quadratures.

Poincaré map

Let γ be a periodic orbit of some flow g^t in $\boldsymbol{R}^{2n}$ arising from a vector field $\overline{f}(x)$. We first take a local cross section $\sum \subset \boldsymbol{R}^n$, of dimension $(2n-1)$. The hypersurface need not be planar, but must be chosen so that the flow is everywhere transverse to it. This is achieved if $\overline{f}(x)\cdot\overline{n}(x)\neq 0$forall $x\in\sum$, where $\overline{n}(x)$ is the unit normal to $\sum$ at x, in Fig. 8.

In our case the normal to the surface, $x_1=0$, is $(1,0,0,0)$ and the tangent to the trajectory is $(\dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2)$ which equals $(y_1, y_2, \dot{y}_1, \dot{y}_2)$ from Hamilton's equations. Thus whenever

$(y_1, y_2, \dot{y}_1, \dot{y}_2)\cdot(1,0,0,0)=0$

or $y_1=0$,

the transversality condition is violated. We impose the additional restriction that $y_1>0$ when $x_1=0$ in order to insure that the transversality condition as $y_1\neq 0$ make the trajectory to pierce the surface section from the "same side". Therefore the surface section on poincaré map is bounded by the curve

$y_1(x_2, y_2; h)=0.$

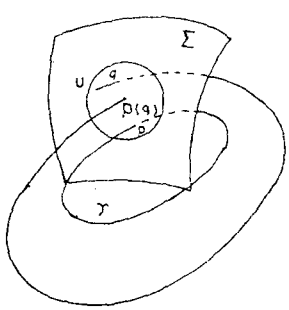


Fig. 8 The poincaré map:
The cross section $\sum=\{x_1=0\}\cap\{H=h\}$

The Poincaré map can be founded analyti cally as follows. Let $F(x,y)=C$ be a first integral which is independent of the energy integral $H=h$. The intersection of the $F=C$ surface with the surface of section $\sum$ represents the invariant curves of the Poincaré map. For fixed energy h, these invariant curves may be written.

$F(x_2, y_2)=F(0, x_2, y_1(x_2, y_2; h), y_2)=C.$

4. *B-G* Transformation

In this section we describe a method that has been developed for constructing approximate formal integrals of a hamiltonian system[8]. The method is a perturbation technique valid for small energies[8, 12]. The method consists of performing a series of canonical transformations which reduces the Hamiltonian to a normal form. For our application, we interest a resonance case only. The resonance means that the frequencies are rationally dependent, that is

$$\sum_{\nu=1}^{n} C_\nu \alpha_\nu = 0 \tag{3}$$

has integer solution C_ν not all equall to zero, i.e., the frequencies α_ν are commensurable[10].

We begin with the system of differential equations

$$\dot{x}=\frac{\partial H}{\partial y}$$

$$\dot{y}=-\frac{\partial H}{\partial x} \tag{4}$$

where x and y are generalized coordinates and mementum vectors, respectively, and where a Hamiltonian H is represented by

$H(x,y)=H^{(2)}(x,y)+H^{(4)}(x,y)+\cdots\cdots.$

The power series of $H(x,y)$ is assumed convergent in neighborhood of $x_1=x_2=0$.

In addition we make the restriction that $H^{(2)}(x, y)$ is positive definite quadratic form. Then we can write H in n-degree-of-freedom

$$H^{(2)}(x, y)=\sum_{\nu=1}^{n}\frac{\alpha_\nu}{2}(x^2+y^2). \tag{5}$$

The Hamiltonian of the system S becomes

$$H(x, y)=H^{(2)}(x, y)+H^{(4)}(x, y)=h \tag{6}$$

where $H(x, y)$ is in the normal form,

and $H^{(4)}(x, y)=\frac{k_1}{4}x_1^4+\frac{k_2}{4}x+{}_2{}^4\frac{k_3}{4}$

$(x_1-x_2)^4$.

We use a first canonical transformation based on the generating function (8) at $s=4$. The assumption (5) means that we have a system of n-uncoupled harmonic oscillators with frequencies $\alpha_\nu, \nu=1, \cdots n$. The higher order terms of $H(x, y)$ have the effect of coupling these oscillators.

The Hamiltonian $H(x, y)$ with $H^{(2)}(x, y)$ given by Ep. (5) is in normal form (3, 5, 10) if $DH(x, y)=0$, where

$$D=\sum_{\nu=1}^{n}\alpha_\nu\left(y_\nu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial y_\nu}\right). \tag{7}$$

The first step is a canonical transformation $(x, y)\rightarrow(\xi, \eta)$ generated by $x\eta+W^{(s)}(x, \eta)$;

$$\xi=x+\frac{W^{(s)}}{\eta}(x, \eta)$$

$$y=\eta+\frac{W^{(s)}}{x}(x, \eta) \tag{8}$$

with $W^{(s)}$ a homogeneous polynomial of degree s such that the new Hamiltonian $\Gamma(\xi, \eta)$ is in normal form upto degree s. Under transformation we have

$$H\left(x, \eta+\frac{W^{(s)}}{x}\right)=\Gamma\left(x+\frac{W^{(s)}}{\eta}, \eta\right) \tag{9}$$

where $\Gamma(\xi, \eta)=\sum_{\nu=2}^{n}\Gamma^{(\nu)}(\xi, \eta)$ is the new Hamiltonian.

We now show that the determining function $W^{(s)}(x, \eta)$ can be chosen so that $\Gamma^{(s)}(\xi, \eta)$ is in normal form. To do this we equate terms of order s in Eq. (9) and obtain

$$DW^{(s)}(x, \eta)=H^{(s)}(x, \eta)-\Gamma^{(s)}(x, \eta) \tag{10}$$

where D is given by Eq. (7). Since $W^{(s)}$, $H^{(s)}$ and $\Gamma^{(s)}$ are homogeneous polynomials of degree s in 2n variables, they can be considered vectors in a space of dimension m given by

$$m=\frac{(2n+s-1)!}{(2n-1)!s!}$$

We consider Eq. (10) as a matrix equation with $W^{(s)}, H^{(s)}$ and $\Gamma^{(s)}$ as m-dimensional column vectors and the linear operator D as a $m\times m$ square matrix. The partial differential operator D admits the decomposition of its domain D into the direct sum of range space R and null space N of D, i.e.,

$D=R\oplus N$ and $R\cap N=\{\phi\}$.

In order to solve the equation (10) we make the canonical transformation

$$x=\frac{1}{\sqrt{2}}(q+ip),\quad \eta=\frac{1}{\sqrt{2}}(q-ip) \tag{11}$$

and express equation (10) in the new complex coordinate $q_\nu, p_\nu, \nu=1, \cdots, n$. The transformation diagonalizes the operator D i.e., $D\tau=i(\alpha, j-k)\tau$ where $\tau=\beta q^j p^k$ and β is a constant. The canonical transformation satisfies the condition which preserves the form of the Hamiltonian equations. Then equations (10) becomes

$$\tilde{D}\tilde{W}^{(s)}(q, p)=\tilde{H}^{(s)}(q, p)-\tilde{\Gamma}^{(s)}(q, p) \tag{12}$$

where $\tilde{D}=\sum_{\nu=1}^{n}i\alpha_\nu\left(q_\nu\frac{\partial}{\partial q_\nu}-p_\nu\frac{\partial}{\partial p_\nu}\right)$.

We note here that the domain of D can be split so that any element of it can be written uniquely as the sum of an element in the null space N, i.e., $(\alpha, j-k)=0$, and an element in the range space R, i.e., $(\alpha, j-k)\neq 0$. We make a unique choice of solution by imposing the condition that $W^{(s)}\in R$. Therefore we can choose $W^{(s)}$ and $\Gamma^{(s)}$ so that $DW^{(s)}=R^{(s)}(x, \eta)$, $W^{(s)}\in R$ and $\Gamma^{(s)}=N^{(s)}$ since $H^{(s)}$ is expressed uniquely as $R^{(s)}(x, \eta)+N^{(s)}(x, \eta)$.

We now split the domain of D as its null space and its range space. The null space

means that the diagonal terms satisfy $a+b-c-d=0$. The values of a, b, c, d are the integer of the power of variables q, p. There are nine diagonal terms of D which vanish in order four. They correspond to the following values of a, b, c, d:

$$(a,b,c,d) = (0,2,0,2)\ (0,2,1,1)\ (0,2,2,0)$$
$$(1,1,0,2)\ (1,1,1,1)\ (1,1,2,0)$$
$$(2,0,0,2)\ (2,0,1,1)\ (2,0,2,0)$$

By Eq. (7) and Eq. (10) we obtain the following expression for the nullspace $N^{(4)}(q,p)$ of $H^{(4)}(q,p)$.

$$N^{(4)}(q,p) = c_1q_2^2p^2{}_2+c_2q_2{}^2p_1p_2+c_3q_2{}^2p_1{}^2+c_4q_1q_2 p_2{}^2+c_5q_1q_2p_1p_2+c_6q_1q_2p_1{}^2+c_7q_1{}^2p_2{}^2 +c_8q_1{}^2p_1p_2+c^9q_1{}^2p_1{}^2 \quad (13)$$

where $c=-\frac{3}{8}(k_2+k_3)$

$$c_2=c_4=c_6=c_8=\frac{3}{4}k_3$$

$$c_3=c_7=-\frac{3}{8}k_3$$

$$c_5=-\frac{3}{2}k_3$$

$$c_9=-\frac{3}{8}(k_1+k_3).$$

Using the inverse of the canonical transformation (11) we transform the equation (13) to the original variables. It this way we find the approximate first integral to be

$$f(x_1,x_2,y_1,y_2) = (k_1+k_3)(x_1{}^2+y_1{}^2)^2 + (k_2+k_3)(x_2{}^2+y_2{}^2)^2 - 4k_3(x_1{}^2+x_2{}^2+y_1{}^2+y_2{}^2) (x_1x_2+y_1y_2)+4k_3(x_1{}^2+y_1{}^2) (x_2{}^2+y_2{}^2)+2k_3(x_1{}^2-y_1{}^2) (x_2-y_2{}^2)+k_3x_1x_2y_1y_2=c \quad (14)$$

The higher order term of $\Gamma^{(i)}$, $i=s+1,\cdots$ are determined uniquely by $H^{(i)}$ and the new specified $W^{(s)}$. We find the term of order (i) as follows.

$$H^{(i)}=\sum\frac{1}{j!}\left\{\frac{\partial^{|j|}\Gamma^{(l)}}{\partial\eta^j}\left(\frac{\partial W^{(s)}}{\partial x}\right)^j - \frac{\partial^{|j|}H^{(l)}}{\partial x^j}\left(\frac{\partial W^{(s)}}{\partial\eta}\right)^j\right\}$$

$$l-|j|+|j|(s-1)=i$$
$$0\le j\le l<i$$
$$l>2,\ s\ge 4 \quad (15)$$

We next solve the equation (15) in order (i) and choose the unique locally generating function $W^{(i)}$ and a new Hamiltonian $\Gamma^{(i)}$ in normal form successively.

In the last step we determine the new integral upto terms of higher order. We perform the inverse transformation that express I in the original coordinates.

The new integral I means an approximate first integral independent of the energy integral, $H=h$, and represents a one-parameter family of invariant curves with C, i.e., I $(x,y)=C$. These curves fill the interior of the region of the x_2-y_2 plane bounded the curve $y_1=0$.

5. Computer Procedure

In these programs the bulk of the work consists in representing, adding, subtracting, multiplying, and differentiating polynomials in 2n variables. We describe here only how to each of these operations is carried out.

In manipulating the homogeneous polynomial we deal only with its coefficients a_j which are stored sequentially in the computer in a lexographic order on the index vector j. We define this order by Exponent mapping. The ordering mapping defines a single valued mapping, say K, of vectors into the positive integers k; i.e.. $k=K(j)$. In order to perform any of the basic polynomial operations, we must also know the inverse mapping $j=K^{-1}(k)$.

Adding and subtracting homogeneous polynomials together is trivial; one merely adds or subtracts corresponding terms in the machine representation of the coefficients.

To multiply

$$\Phi^{(\nu)}=\sum_{|j|=\nu}a_jx^j \quad \text{by} \quad \Psi^{(\eta)}(x\cdots,x_i)=\sum_{|l|=\eta}b_lx^l$$

We perform a double sum over j and l : given $j=(j_1,\cdots,j_4)$ and $l=(l_1,\cdots l_4)$ we form $j+l=(j_1+l_1,\cdots,j_4+l_4)$ and than find the unique integer k associated it. The partial product a_jb_l is then added into location k of the product. Finally, to compute the partial derivative of $\Phi^{(\nu)}$ with respect to x_σ, $1\leq\sigma\leq4$, we search out in increasing order over k, using $j=K^{-1}(k)$, those vectors j which have a nonzero component j_σ. For each of these vectors we form the product $j_\sigma a_j$ and store it in the same sequence as it was found, which happens to be the correct machine representation. These products then constitute the coefficients of $\partial\Phi^{(\nu)}/\partial x_\sigma$ · The integer k becomes $(l_1,\cdots,l_\sigma-1,\cdots,l_4)$, $1\leq\sigma\leq4$.

The exponent mapping is represented by the order of integer as follows

$\mathrm{Exp}(i_1,i_2,j_1,j_2)$ for $0\leq i\leq8, 0\leq j\leq8, s=i+j$.

The canonical transformation is represented by $C_{n,\nu}i^\nu q^\nu\ p^{n-\nu}$, where $C_{n,\nu}=n!/\nu!\,(n-\nu)!$. The '$E$' operation in computer is simply the summation of the integer power $\alpha(i-j)$. Then we make a matrix space in the operation D, i.e., $a(m,n)=\alpha_\nu i_\nu$, where $m=k$, $n=K(i_\nu-1,j_\nu+1)$.

6. Experimental Results

We procedure the computer program applied by the Poincaré map. The program performed $B-G$ transformation repeatedly upto eight order polynomlals. Then we investigate the system S in more detail than Month and Rand.

To investigate the system S in symmetric system, $k_1=k_2=k$, the results are shown in Fig. 11 (a) and

(1) In-phase mode: stable before bifurcation $k>0$,
unstable after bifurcation $k<0$. $(k_3>0)$

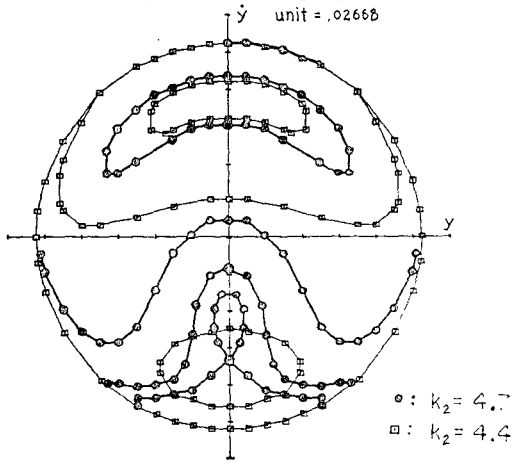
(2) Out-phase mode: stable before bifurcation $k<4k_3$,
unstable after bifurcation $k>4k_3$ $(k_3>0)$

In unsymmetric system, $k_1\neq k_2$ at fixed $k_3=1$, the results are shown in Fig. 11 (b).

(1) In-phase mode: unstable in the dark solid lines.
$(2k_1<k_2<1/2k_1,\ 0<k_1,\ k_2<0,\ k_3<0)$

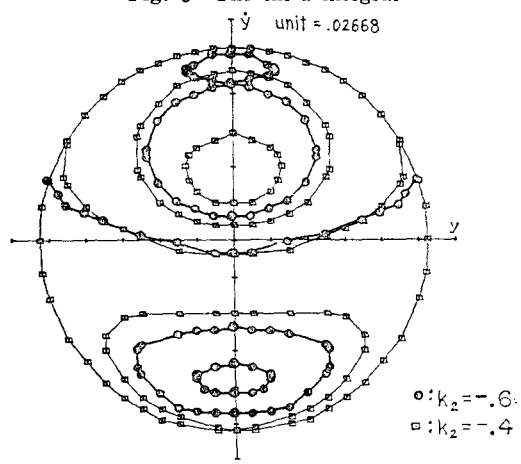
(2) Out-phase mode: unstable in the dark solid lines.
$(4k_3+1/2k_1<k_2<4k_3+2k_1,\ k_1,k_2,k_3>0)$

In Fig. 9, 10 we know the boundary of bifurcations at k_3 fixed in k_1-k_2 space. As you see, modes are bifurcated from stable to unstable when k_1 changes slightly or k_2. Out-of-



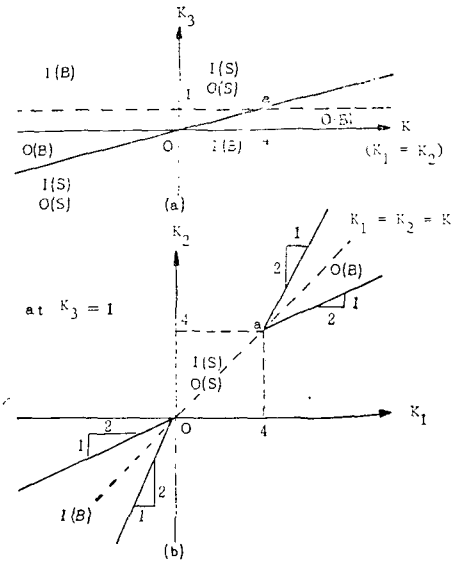
$k_1=5$, $k_3=1$, $h=.01$

Fig. 9 The third integral

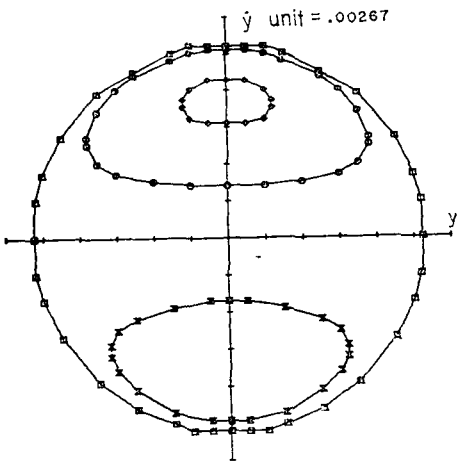


$k_1=-1$, $k_3=1$, $h=.01$

Fig. 10 The third integral



(a) the symmetric system ($K=K_1=K_2$)
(b) the unsymmetric system ($K_1\neq K_2$), at $K_3=1$)
Fig. 11 The parametric spaces for the system S

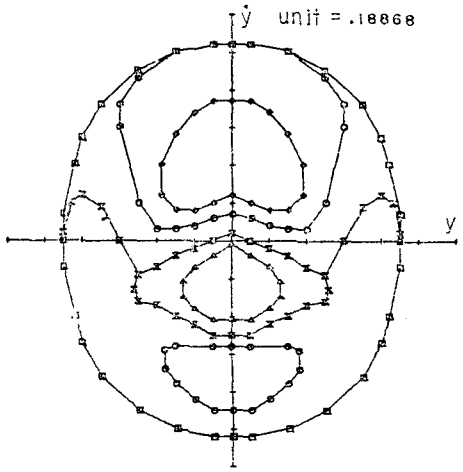


$k_1=k_2=.0$, $k_3=1$, $h=.0001$
Fig. 12 The third integral

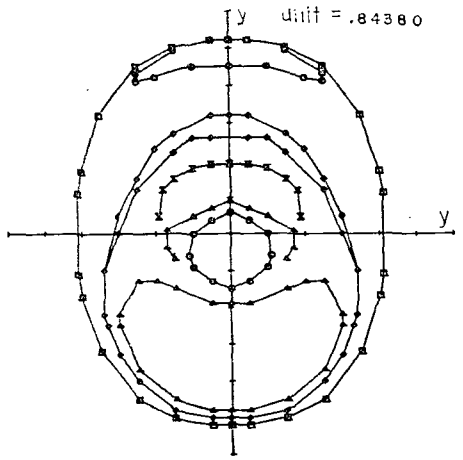
phase mode represents in Fig. 9 and in-phase mode in Fig. 10.

We investigate the dynamical structure of bifurcating points, $k_1=k_2=0$ and $k_1=k_2=4$ at $k_3=1$, respectively.

For small energy both phase modes are stable represented by concentric circles. When h increase, the NNM's proceed to the bifurcation.



$k_1=k_2=.0$, $k_3=1$, $h=.5$
Fig. 13 The third integral



$k_1=k_2=.0$, $k_3=1$, $h=10$
Fig. 14 The third integral

At last the motion tends to be ergodic as h becomes large.

7. Conclusions

A computer program has been here constructed and named as TIC 4 to find the third integral with the higher order terms by performing a series of the Birkhoff-Gustavson transformation and we investigate the dynamical structure of nonlinear coupled oscillator S by performing the computer procedure (TIC-4) and generating an approximation for the Poin-

caré map.

The system S consists of two unit masses supported by nonlinear springs whose linearized frequencies are in 1 : 1 resonance and the potential energy consists of quadratic and 4th order terms of displacement. It is that the system S admits only 2 or 4 NNM's, depending upon the values of the parameter k_1, k_2, and k_3. It is also found that the bifurcating modes enter as stable while the mode from which they bifurcated changes from stable to unstable upon bifurcation.

It is emphasized that all these results are valid only for small h. The reason is that KAM (Kolmogorov-Arnold-Moser) theory tells us that invariant tori generally do not fill the energy manifold $H=h$ in systems which are not integrable. For large enough energy h the Poincaré map appears to be filled with "noise", i.e., which motions seem to be ergodic.

The computer procedure becomes better as the order incrases, but it is an open question whether the integral is a convergent or an asymptotic representation of the integral surfaces. However, it is assumed here that the polynomials upto eight converges to the integral of motion. It is important to mention the amount of time that problems consume in obtaining the data points for the level curves. By using TIC-4 program for the system S, it took average 3.50 min to compute the normal form and 5.62 min. to perform the graphic procedure, while the forward integral (Runge-Kutta method) took 8 min. to calculate a single initial condition with a step size of 0.0005.

It is also noted that although we investigated the system S in 1 : 1 resonance by B-G transformation and the Poincaré map, it can be obtained another system in resonance by similar methods.

Finally, we note that for determining the stability of NNM's in autonomous two-degree-of-freedom Hamiltonian system S, the integrals and the Poincaré map can predict accurate dynamical behavior near an equilibrium point.

The computer procedure, the TIC-4, save the mount of C.P.U. time largely than the forward integral.

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Appendix

Definition: For a System $\ddot{x}+\frac{\partial v}{\partial x}=0$, $x=(x_1,\cdots x_n)$.

(1) Every component $x_i(t)$, $i=1,\cdots n$ has the same frequency;

(2) There exists some t such that $x_i(\bar{t})=0$ for $i=1,\cdots n$, that is, a $\bar{t}$ for which all of the componants vanish simultaneously;

(3) All of the component functions $x_i(t)$ take on their extreme values at the same time;

(4) For every t and for every i, $i=1,\cdots,n$, the value $x_1(\bar{t}),\cdots,x_{i-1}(\bar{t})$, $x_{i+1}(\bar{t}),\cdots,x_n(\bar{t})$ are single valued functions of $x_i(t)$.