

TRANSLATION THEOREMS FOR FEYNMAN INTEGRALS ON ABSTRACT WIENER AND HILBERT SPACES

DONG MYUNG CHUNG AND SOON JA KANG

1. Introduction

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let m be the Gauss measure on H defined by

$$m(A) = (2\pi)^{-n/2} \int_F \exp \left\{ -\frac{|x|^2}{2} \right\} dx,$$

where $A=P^{-1}(F)$, F is a Borel set in the image of an n -dimensional projection P in H and dx is Lebesgue measure in PH . A norm $\| \cdot \|$ on H is called *measurable* if for every $\varepsilon > 0$ there exists a finite dimensional projection P_0 such that $m(\{x \in H: \|Px\| > \varepsilon\}) < \varepsilon$ whenever P is a finite dimensional projection orthogonal to P_0 . It is known (see [7]) that H is not complete with respect to $\| \cdot \|$. Let B denote the completion of H with respect to $\| \cdot \|$. Let i denote the natural injection from H into B . The adjoint operator i^* is one-to-one and maps B^* continuously onto a dense subset of H^* . By identifying H^* with H and B^* with i^*B^* , we have a triple $B^* \subset H \subset B$ and $\langle x, y \rangle = (x, y)$ for all x in H and y in B^* , where (\cdot, \cdot) denote the natural dual pairing between B and B^* . By a well-known result of Gross, $m \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ -algebra $\mathcal{B}(B)$ of B . The triple (H, B, ν) is called an *abstract Wiener space*. For more details, see Kuo [7].

In [5], Kallianpur and Bromley defined analytic Feynman integrals on an abstract Wiener space (H, B) , and established the existence of the integrals for integrands belonging to a Fresnel class $\mathcal{F}(B)$ of functionals on (H, B) , which is the extension of the results on analytic Feynman integrals on Wiener space which Cameron and Storvick obtained in [2]. Recently, in [6] Kallianpur, Kannan and Karandikar defined sequential

Received May 19, 1986.

Supported by a grant from the Ministry of Education (1986).

Feynman integrals on an abstract Wiener and Hilbert spaces and established the existence of both of analytic and sequential Feynman integrals for integrands belonging to larger classes $\mathcal{Q}^q(B)$ and $\mathcal{Q}^q(H)$ than Fresnel classes $\mathcal{F}(B)$ and $\mathcal{F}(H)$ considered in Kallianpur and Bromley [5] and Albeverio and Hoegh-Krohn [1].

The purpose of this paper is, by using Cameron and Storvick's proof in [3], to prove translation theorems for both of analytic and sequential Feynman integrals for the classes $\mathcal{Q}^q(B)$ and $\mathcal{Q}^q(H)$ considered in [6]. These results are generalizations of the results on translation theorems for analytic and sequential Feynman integrals on Wiener space in [3] and [4], and for Fresnel integrals on Hilbert spaces in [1].

2. Preliminaries

Let (H, B, ν) be an abstract Wiener space. Let \mathcal{D} denote the ordered set of all finite dimensional orthogonal projections P of H ($P < Q$ if $PH \subseteq QH$). For $P \in \mathcal{D}$, let

$$C_P = \{P^{-1}(F) : F \text{ is a Borel set of } PH\} \text{ and } \mathcal{R}_H = \bigcup_P C_P.$$

Then it is easy to see that \mathcal{R}_H is an algebra of subsets of H .

Let $\{e_j, j \geq 1\}$ be a complete orthonormal system in H such that e_j 's are in B^* . For each h in H and x in B , let

$$(h, x)^\sim = \begin{cases} \lim_{n \rightarrow \infty} \sum_{j=1}^n \langle h, e_j \rangle \langle x, e_j \rangle, & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(h, \cdot)^\sim$ is a Borel measurable functional on B and if both h and x are in H , Parseval's identity gives $(h, x)^\sim = \langle h, x \rangle$ (see [5]). A function f in H of the form $f(h) = \phi(\langle h_1, h \rangle, \dots, \langle h_k, h \rangle)$ is called a *cylinder function* on H , where $h_i \in H$, and ϕ is a Borel function on \mathbf{R}^k . We denote by $R(f)$ the random variable $\phi((h_1, x)^\sim, \dots, (h_k, x)^\sim)$ on B .

DEFINITION 2.1. Let $L(H, \mathcal{R}_H, m)$ be the class of complex-valued continuous functions f on H such that the net $\{R(f \circ P) : P \in \mathcal{D}\}$ is Cauchy in ν -probability. Further, for $f \in L(H, \mathcal{R}_H, m)$, let

$$R(f) = \lim_{P \in \mathcal{D}} R(f \circ P) \text{ in } \nu\text{-probability.}$$

The mapping R is called an *m-lifting* (see [6]).

DEFINITION 2.2. Let

$$L^1(H, \mathcal{R}_H, m) = \{f \in L(H, \mathcal{R}_H, m) : \int_B |R(f)| d\nu < \infty\}$$

and for $f \in L^1(H, \mathcal{R}_H, m)$, define

$$\int_H f dm = \int_B R(f) d\nu.$$

Let $\mathcal{M}(H)$ be the class of all countably additive complex measures on Borel subsets of H with finite absolute variation. Let $\mathcal{F}(H)$ be the class of all functions f of the form

$$(2.1) \quad f(h) = \int_H e^{i\langle h, h_1 \rangle} d\mu(h_1), \quad h \in H,$$

for some $\mu \in \mathcal{M}(H)$. $\mathcal{F}(H)$ is called the *Fresnel class* of functions on H . Let $\theta : B \rightarrow \mathbf{C}$. For $\lambda > 0$, we denote θ^λ by the function defined by $\theta^\lambda(x) = \theta(\lambda^{-1/2}x)$, $x \in B$.

LEMMA 2.1 [6]. Let $f \in \mathcal{F}(H)$ be of the form as (2.1). Then $f \in L^1(H, \mathcal{R}_H, m)$ and $R(f) = F$, where F is given by

$$(2.2) \quad F(x) = \int_H e^{i\langle h, x \rangle} d\mu(h), \quad x \in B.$$

and for $\lambda > 0$, we have $R(f^\lambda) = F^\lambda$ for all $\lambda > 0$.

Further, if $P_n \xrightarrow{s} I$ (i.e. P_n converges strongly to the identity operator I), then we have $R(f^\lambda \circ P_n) \rightarrow R(f^\lambda) = F^\lambda$ in $L^1(B, \mathcal{B}(B), \nu)$.

LEMMA 2.2 [6]. Let A be a self adjoint trace class operator with eigen values α_k and corresponding eigenfunctions $\{e_k\}$. Let $u(h) = \langle h, Ah \rangle$, $h \in H$. Then, for all $\lambda > 0$, $u^\lambda \in L(H, \mathcal{R}_H, m)$ and $R(u^\lambda) = v^\lambda$, where v is given by

$$v(x) = \begin{cases} \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k [e_k, x]^\sim{}^2, & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

and v is denoted by $(x, Ax)^\sim$.

LEMMA 2.3 [6]. Let $\mu \in \mathcal{M}(H)$ and A be a self adjoint trace class operator on H . Let g, G be defined by

$$(2.3) \quad g(h) = e^{i/2 \langle h, Ah \rangle} \int_H e^{i\langle h_1, h \rangle} d\mu(h_1)$$

and

$$(2.4) \quad G(x) = e^{i/2 \langle x, Ax \rangle} \int_H e^{i\langle h, x \rangle} d\mu(h) = e^{i/2 \langle x, Ax \rangle} F(x), \quad \text{say.}$$

then for $\lambda > 0$, we have $R(g^\lambda) = G^\lambda$ and further if $P_n \xrightarrow{s} I$, then

$$R(g^{\lambda} \circ P_n) \rightarrow G^{\lambda} \text{ in } L^1(B, \mathcal{B}(B), \nu).$$

For a real number q , $q \neq 0$, let $\mathcal{Q}^q(H)$ [resp. $\mathcal{Q}^q(B)$] denote the class of functions g [resp. G] defined by (2.3) [resp. (2.4)] for some $\mu \in \mathcal{M}(H)$ and some self adjoint trace class operator A on H such that the bounded inverse $(I+1/qA)^{-1}$ exists.

For a self adjoint trace class operator A with eigenvalues $\{\alpha_j\}$, the Fredholm determinant of $(I+A)$ (denoted by $\det(I+A)$) is defined by $\det(I+A) = \prod_{j=1}^{\infty} (1+\alpha_j)$, and the Maslov index of $(I+A)$ (denoted by $\text{ind}(I+A)$) is the number of negative eigenvalues of $(I+A)$, i.e. $\text{ind}(I+A) = \# \{j : 1+\alpha_j < 0\}$.

3. Translation theorems for analytic and sequential Feynman integrals on B .

In this section, we prove translation theorems for both of analytic and sequential Feynman integrals for the class $\mathcal{Q}^q(B)$.

DEFINITION 3.1. Let F be a measurable complex-valued function on B such that

$$(i) \quad J_F(\lambda) = \int_B F(\lambda^{-1/2}x) d\nu(x)$$

exists for all real $\lambda > 0$.

(ii) There is an analytic function J_F^* on $\Omega = \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all real $\lambda > 0$. Then for $z \in \Omega$, $I_a^z(F) \equiv J_F^*(z)$ is called the *analytic Wiener integral* of F over B with parameter z . For a real q ($q \neq 0$), if the limit

$$\lim_{\substack{z \rightarrow -iq \\ z \in \Omega}} I_a^z(F) = I_a^q(F)$$

exists, $I_a^q(F)$ is called the *analytic Feynman integral* of F over B with parameter q .

Given two complex-valued functions F and G on B , F is said to be equal to G *s-almost surely* (s-a.s.) if for each $\alpha > 0$, $\nu\{x \in B : F(\alpha x) \neq G(\alpha x)\} = 0$. Let's denote this equivalence relation between functions on B by $F \approx G$ or $F(x) \approx G(x)$, $x \in B$. It is easy to see that if $F \approx G$, then $J_F(\lambda) = J_G(\lambda)$ for all real $\lambda > 0$. For a function F on B , let $[F]$ denote the equivalence class of functionals which are equal to F s-a.s.. The class of equivalence classes defined by

$$\mathcal{F}(B) = \{[F]; F(x) = \int_H e^{i\langle h, x \rangle} d\mu(h), \mu \in \mathcal{M}(H)\}$$

is called the *Fresnel class* of functions on B . It is known [5] that $\mathcal{F}(B)$ forms a Banach algebra over the complex field. As is customary, we will identify a function with its s -equivalence class and think of $\mathcal{F}(B)$ as a class of functions on B rather than as a class of equivalence classes.

DEFINITION 3.2. Let F be a measurable complex-valued function on B such that $R(f^\lambda) = F^\lambda$ for all $\lambda > 0$ for some $f \in L(H, \mathcal{Q}_H, m)$ and let $F_P = R(f \circ P)$ for all $P \in \mathcal{D}$. Suppose that $I_a^z(F_P)$ exists for all $P \in \mathcal{D}$ and that the limit $\lim_{n \rightarrow \infty} I_a^{z_n}(F_{P_n}) = I_s^q(F)$ exists for all $z_n \rightarrow -iq (q \neq 0)$, $z_n \in \Omega$ and for all $P_n \xrightarrow{s} I$, $P_n \in \mathcal{D}$. Then $I_s^q(F)$ is called the *sequential Feynman integral* of F over B with parameter q .

THEOREM 3.1 [6]. Let $\mu \in \mathcal{M}(H)$ and let A be a self adjoint trace class operator such that $\left(I + \frac{1}{q}A\right)^{-1} (q \neq 0)$ exists, let $G \in \mathcal{Q}^q(B)$ be given by

$$G(x) = e^{\frac{i}{2}\langle x, Ax \rangle} \int_H e^{i\langle h, x \rangle} d\mu, \quad x \in B.$$

Then $I_a^q(G)$ and $I_s^q(G)$ exist and

$$I_a^q(G) = I_s^q(G) = \left| \det \left(I + \frac{1}{q}A \right) \right|^{-1/2} e^{-\frac{i\pi}{2} \text{ind} \left(I + \frac{1}{q}A \right)} \cdot \int_H e^{-\frac{i}{2q} \langle \left(I + \frac{1}{q}A \right)^{-1} h, h \rangle} d\mu(h).$$

LEMMA 3.2. Let K and L be complex-valued measurable functions on B . If $K \approx L$ and $h \in H$, then $K(\cdot + h) \approx L(\cdot + h)$.

Proof. Let $\lambda > 0$ be given. Since $K \approx L$, $E_\lambda = \{x \in B : K(\lambda x) \neq L(\lambda x)\}$ is a ν -null set. Let I_{E_λ} denote the indicator function of E_λ on B . Then, since $\nu(E_\lambda) = 0$, it follows from [7] that

$$\int_B I_{E_\lambda} \left(x + \frac{h}{\lambda} \right) d\nu(x) = e^{-\frac{1}{2\lambda} |h|^2} \int_B I_{E_\lambda}(x) e^{i\langle \frac{h}{\lambda}, x \rangle} d\nu(x) = 0.$$

Hence $I_{E_\lambda} \left(x + \frac{h}{\lambda} \right) = 0$ for almost all $x \in B$, so that $x + \frac{h}{\lambda} \notin E_\lambda$ for almost all $x \in B$. Therefore, we have $K(\cdot + h) \approx L(\cdot + h)$.

THEOREM 3.3. Let $y \in H$, and let $G \in \mathcal{Q}^q (q \in \mathbf{R}, q \neq 0)$ be given by

$$G(x) = e^{\frac{i}{2}\langle x, Ax \rangle} F(x), \quad x \in B$$

where for some $\mu \in \mathcal{M}(H)$,

$$F(x) \approx \int_H e^{i(h, x) \sim} d\mu(h), \quad x \in B.$$

Then $K(\cdot) = G(\cdot + y)$ is in the class $\mathcal{Q}^q(B)$, and

$$I_a^q(K) = I_s^q(K) = |\det(I + \frac{1}{q}A)^{-1}|^{-1/2} e^{-\frac{i\pi}{2} \text{ind}(I + \frac{1}{q}A)} \\ e^{\frac{i}{2} \langle y, Ay \rangle} \int_H e^{-\frac{i}{2q} \langle (I + \frac{1}{q}A)^{-1}(h + Ay), (h + Ay) \rangle} e^{i \langle h, y \rangle} d\mu(h).$$

Proof. Since

$$G(x) \approx e^{\frac{i}{2} \langle x, Ax \rangle \sim} \int_H e^{i(h, x) \sim} d\mu(h), \quad x \in B,$$

it follow from Lemma 3.2 that

$$G(x+y) \approx e^{\frac{i}{2} \langle x+y, A(x+y) \rangle \sim} \int_H e^{i(h, x+y) \sim} d\mu(h), \quad x \in B \\ = e^{\frac{i}{2} \langle x, Ax \rangle \sim} \cdot \int_H e^{i(h+Ay, x) \sim} \cdot e^{\frac{i}{2} \langle y, Ay \rangle} \cdot e^{i \langle h, y \rangle} d\mu(h) \\ = e^{\frac{i}{2} \langle x, Ax \rangle \sim} \int_H e^{i(h, x) \sim} d\mu^*(h),$$

where for any Borel set E in H ,

$$\mu^*(E) = \tilde{\mu}(E - Ay) \quad \text{and} \quad \tilde{\mu}(E) = e^{\frac{i}{2} \langle y, Ay \rangle} \int_E e^{i \langle h, y \rangle} d\mu(h).$$

Clearly, we have $\mu^* \in \mathcal{M}(H)$ and $K(\cdot) = G(\cdot + y) \in \mathcal{Q}^q(B)$. Therefore, by Theorem 3.1, we have

$$I_a^q(K) = I_s^q(K) \\ = |\det(I + \frac{1}{q}A)^{-1}|^{-1/2} e^{-\frac{i\pi}{2} \text{ind}(I + \frac{1}{q}A)} \cdot \int_H e^{-\frac{i}{2q} \langle (I + \frac{1}{q}A)^{-1}h, h \rangle} d\mu^*(h) \\ = |\det(I + \frac{1}{q}A)^{-1}|^{-1/2} e^{-\frac{i\pi}{2} \text{ind}(I + \frac{1}{q}A)} \\ e^{\frac{i}{2} \langle y, Ay \rangle} \int_H e^{-\frac{i}{2q} \langle (I + \frac{1}{q}A)^{-1}(h + Ay), (h + Ay) \rangle} \cdot e^{i \langle h, y \rangle} d\mu(h),$$

completing the proof of theorem.

THEOREM 3.4. Let $G \in \mathcal{Q}^q(B)$ ($q \in \mathbf{R}$, $q \neq 0$) be given by

$$G(X) = \exp \{i/2 \langle x, Ax \rangle \sim\} F(x),$$

where $F \in \mathcal{F}(B)$. Then for each $y \in H$, we have

$$I_a^q(G) = e^{\frac{qi}{2} |y|^2} I_a^q(G(\cdot + y) \cdot e^{iq(y, \cdot) \sim}),$$

$$I_s^q(G) = e^{\frac{qi}{2} |y|^2} I_s^q(G(\cdot + y) \cdot e^{iq(y, \cdot) \sim}).$$

Proof. Since $F \in \mathcal{F}(B)$, for some $\mu \in \mathcal{M}(H)$,

$$F(x) \approx \int_H e^{i\langle h, x \rangle} d\mu(h), \quad x \in B.$$

Thus we have

$$\begin{aligned} L(x) &\equiv G(x) e^{-iq\langle y, x \rangle} \approx e^{\frac{i}{2}\langle x, Ax \rangle} \int_H e^{i\langle h, x \rangle} e^{-iq\langle y, x \rangle} d\mu(h), \quad x \in B \\ &= e^{\frac{i}{2}\langle x, Ax \rangle} \int_H e^{i\langle h - qy, x \rangle} d\mu(h) \\ &= e^{\frac{i}{2}\langle x, Ax \rangle} \int_H e^{i\langle h, x \rangle} d\mu^*(h), \end{aligned}$$

where $\mu^*(E) = \mu(E + qy)$ for $E \in \mathcal{B}(H)$. Clearly, we have $\mu^* \in \mathcal{M}(H)$ and $L \in \mathcal{Q}^q(B)$. Hence by Theorem 3.1,

$$\begin{aligned} I_a^q(L) &= |\det(I + \frac{1}{q}A)|^{-1/2} e^{-\frac{\pi}{2}\text{ind}(I + \frac{1}{q}A)} \cdot \int_H e^{\frac{-i}{2q}\langle (I + \frac{1}{q}A)^{-1}h, h \rangle} d\mu^*(h) \\ &= |\det(I + \frac{1}{q}A)|^{-1/2} e^{-\frac{\pi}{2}\text{ind}(I + \frac{1}{q}A)} \cdot \int_H e^{\frac{-i}{2q}\langle (I + \frac{1}{q}A)^{-1}(h - qy), (h - qy) \rangle} d\mu(h). \end{aligned}$$

To evaluate the integral on the left hand side of the above equation, we observe that

$$\begin{aligned} &\langle (I + \frac{1}{q}A)^{-1}(h - qy), (h - qy) \rangle \\ &= \langle (I + \frac{1}{q}A)^{-1}(h + Ay - qy - Ay), h + Ay - qy - Ay \rangle \\ &= \langle (I + \frac{1}{q}A)^{-1}[(h + Ay) - q(I + \frac{1}{q}A)y], (h + Ay) - q(I + \frac{1}{q}A)y \rangle \\ &= \langle (I + \frac{1}{q}A)^{-1}(h + Ay) - qy, (h + Ay) - q(I + \frac{1}{q}A)y \rangle \\ &= \langle (I + \frac{1}{q}A)^{-1}(h + Ay), h + Ay \rangle - q\langle y, Ay \rangle - 2q\langle y, h \rangle + q^2\langle y, y \rangle. \end{aligned}$$

Thus we have

$$\begin{aligned} &\int_H e^{\frac{-i}{2q}\langle (I + \frac{1}{q}A)^{-1}(h - qy), h - qy \rangle} d\mu(h) \\ &= e^{-\frac{qi}{2}\|y\|^2} e^{\frac{i}{2}\langle y, Ay \rangle} \int_H e^{\frac{-i}{2q}\langle (I + \frac{1}{q}A)^{-1}(h + Ay), h + Ay \rangle} e^{i\langle y, h \rangle} d\mu(h). \end{aligned}$$

Hence, by Lemma 3.2 and Theorem 3.3,

$$\begin{aligned} I_a^q(H) &= I_a^q(G(\cdot) e^{-iq\langle y, \cdot \rangle}) = I_s^q(G(\cdot) e^{-iq\langle y, x \rangle}) \\ &= e^{-\frac{qi}{2}\|y\|^2} \cdot I_a^q(G(\cdot + y)) = e^{-\frac{qi}{2}\|y\|^2} \cdot I_s^q(G(\cdot + y)). \end{aligned}$$

This completes the proof of the theorem.

REMARK. Taking A to be the zero operator in Theorem 3.4, we obtain the results on translation theorems for both of analytic and sequential Feynman integrals for the Fresnel class of functions on B (cf. [3], [4]).

4. Translation theorems for analytic and sequential Feynman Integrals on H .

In this section we prove translation theorems for both of analytic and sequential Feynman integrals for the class $\mathcal{Q}^q(H)$ and show how those integrals can be modified so as to be translation invariant.

Let $f: H \rightarrow \mathbb{C}$ be such that for all real $\lambda > 0$, $f^\lambda \in L^1(H, \mathcal{R}_H, m)$. For real $\lambda > 0$, let

$$K_f(\lambda) = \int_H f^\lambda dm.$$

DEFINITION 4.1. Let f be such that there exists an analytic function $K_f^*(z)$ on Ω such that $K_f^*(\lambda) = K_f(\lambda)$ for all real $\lambda > 0$. Then for $z \in \Omega$, $K_f^*(z) \equiv I_a^z(f)$ is called the analytic Gauss integral of f over H with parameter z . For a real q ($q \neq 0$), if the limit

$$\lim_{\substack{z \rightarrow -iq \\ z \in \Omega}} I_a^z(f) = I_a^q(f)$$

exists, $I_a^q(f)$ is called the analytic Feynman integral of f over H with parameter q .

Let $f: H \rightarrow \mathbb{C}$ be such that for all $P \in \mathcal{P}$, for all real $\lambda > 0$,

$$(4.1) \quad \int_{\mathbb{R}^m} |f(\sum_{j=1}^m \xi_j e_j')| e^{-\frac{\lambda}{2} \sum_{j=1}^m \xi_j^2} d\xi < \infty$$

where $m = \dim PH$ and $\{e_1', e_2', \dots, e_m'\}$ is an orthonormal basis for PH and for $z \in \Omega$, define

$$J_f(z, P) = \left[\left(\frac{z}{2\pi} \right)^{1/2} \right]^m \int_{\mathbb{R}^m} f(\sum_{j=1}^m \xi_j e_j') e^{-\frac{z}{2} \sum_{j=1}^m \xi_j^2} d\xi.$$

DEFINITION 4.2. Let f satisfy (4.1) for all $\lambda > 0$ and $P \in \mathcal{P}$. Suppose that the limit

$$\lim_{n \rightarrow \infty} J_f(z_n, P_n) = I_s^q(f)$$

exists for all $z_n \rightarrow -iq$, $z_n \in \Omega$ and for all $P_n \xrightarrow{s} P$, $P_n \in \mathcal{P}$. The limit $I_s^q(f)$, easily seen to be independent of $\{z_n\}$, $\{P_n\}$, is called the sequential Feynman integral of f over H with parameter q .

THEOREM 4.1 [6]. Let $\mu \in \mathcal{M}(H)$ and let A be a self adjoint trace class operator such that $\left(I + \frac{1}{q}A\right)^{-1}$ exists, ($q \in \mathbf{R}$, $q \neq 0$). Let $g \in \mathcal{Q}^q(H)$ be given by

$$g(h) = e^{\frac{i}{2}\langle h, Ah \rangle} \int_H e^{i\langle h_1, h \rangle} d\mu(h_1), \quad h \in H.$$

Then $I_a^q(g)$ and $I_s^q(g)$ exist and

$$I_a(g) = I_s^q(g) = |\det(I + \frac{1}{q}A)|^{-1/2} e^{-\frac{i\pi}{2} \text{ind}(I + \frac{1}{q}A)} \int_H e^{-\frac{i}{2q}\langle h, (I + \frac{1}{q}A)^{-1}h \rangle} d\mu(h).$$

THEOREM 4.2. Let $g \in \mathcal{Q}^q(H)$, ($q \in \mathbf{R}$, $q \neq 0$) and $y \in H$. Then $g(\cdot + y) \in \mathcal{Q}^q(H)$ and

$$I_a^q(g(\cdot + y)) = e^{\frac{qi}{2}|y|^2} I_a^q(g \cdot e^{-iq\langle y, \cdot \rangle})$$

$$I_s^q(g(\cdot + y)) = e^{\frac{qi}{2}|y|^2} I_s^q(g \cdot e^{-iq\langle y, \cdot \rangle}).$$

Proof. The proof of the theorem can be shown by using Theorem 4.1 and the same argument as was used in Theorem 3.3 and Theorem 3.4.

COROLLARY 4.3. Under the hypothesis of Theorem 4.2, we have $g(\cdot + y) \in \mathcal{Q}^q(H)$ and

$$I_a^q(g) = e^{\frac{qi}{2}|y|^2} \cdot I_a^q(g(\cdot + y) \cdot e^{iq\langle y, \cdot \rangle}),$$

$$I_s^q(g) = e^{\frac{qi}{2}|y|^2} \cdot I_s^q(g(\cdot + y) \cdot e^{iq\langle y, \cdot \rangle}).$$

Following Cameron and Storvick [4], we define the translation invariant analytic and sequential Feynman integrability of function on H .

DEFINITION 4.3. Let g be analytic [resp. sequential] Feynman integrable over H for some real $q \neq 0$. Let

$$(4.2) \quad K(h) \equiv \exp\left\{\frac{iq}{2}|h|^2\right\} g(h), \quad h \in H.$$

Then we define the translation invariant analytic [resp. sequential] Feynman integral of K over H with parameter q by

$$(4.3) \quad I_a^{iq}(K) = I_a^q(g) \quad [\text{resp.} \quad I_s^{iq}(K) = I_s^q(g)].$$

THEOREM 4.4. Let q be real, $q \neq 0$, let $g \in \mathcal{Q}^q(H)$, and let K be given by (4.2). Then for each $y \in H$, $K(\cdot + y)$ is translation invariant analytic and sequential Feynman integrable over H with parameter q , and

$$I_a^{iq}(K(\cdot + y)) = I_a^{iq}(K), \text{ and } I_s^{iq}(K(\cdot + y)) = I_s^{iq}(K).$$

Proof. By (4.2) we have, for $h \in H$

$$\begin{aligned} K(h+y) &= \exp \left\{ \frac{iq}{2} |h+y|^2 \right\} g(h+y) \\ &= \exp \left\{ \frac{iq}{2} |h|^2 \right\} \cdot \exp \{iq \langle h, y \rangle\} \exp \left\{ \frac{iq}{2} |y|^2 \right\} g(h+y). \end{aligned}$$

By Theorem 4.2, $g(\cdot + y) \in \mathcal{Q}^q(H)$ and so

$$g(h+y) = e^{\frac{i}{2} \langle h, Ah \rangle} \int_H e^{i \langle h_1, h \rangle} d\mu(h_1),$$

where $\mu \in \mathcal{M}(H)$ and A is a self adjoint trace class operator such that $(I + \frac{1}{q}A)^{-1}$ exists. Now, let

$$\tilde{\mu}(E) = e^{\frac{i}{2} |h|^2} \int_E e^{i \langle h_1, h \rangle} d\mu(h_1 - qy)$$

where E is a Borel set in H . Then we can write

$$K(h+y) = e^{\frac{iq}{2} |h|^2} \cdot e^{\frac{i}{2} \langle h, Ah \rangle} \cdot \int_H e^{i \langle h_1, h \rangle} d\tilde{\mu}(h_1), \quad h \in H.$$

Hence $K(\cdot + y)$ is translation invariant analytic and sequential Feynman integrable. Applying Corollary 4.3 and equations (4.2) and (4.3). we have

$$\begin{aligned} I_a^{iq}(K(\cdot + y)) &= I_a^q(e^{iq \langle \cdot, y \rangle} \cdot e^{\frac{iq}{2} |y|^2} g(\cdot + y)) \\ &= I_a^q(g) = I_a^{iq}(K) \end{aligned}$$

and similarly we have $I_s^{iq}(K(\cdot + y)) = I_s^{iq}(K)$, completing the proof of the theorem.

References

1. A. Alberverio and R. Hoegh-Krohn, *Mathematical theory of Feynman path integrals*, Lecture Notes in Mathematics, No. 523. Springer-Verlag, Berlin (1976).
2. R.H. Cameron and D.A. Storvick, *Some Banach algebras of analytic Feynman integrable functionals*. *Analytic Functions*, Kozubnik 1979, Lecture notes in Mathematics, No. 798, Springer-Verlag, Berlin (1980), pp.18-67.

3. R.H. Cameron and D.A. Storvick, *A new translation theorem for the analytic Feynman integral*, Rev. Roumaine Math. Pures Appl. Vol. **27**, No. 9(1982), 937-944.
4. R.H. Cameron and D.A. Storvick, *A simple definition of the Feynman integral with applications*, Memoirs of the American Mathematical Society, No. 288, (1983).
5. G. Kallianpur and C. Bromley, *Generalized Feynman integrals using analytic continuation in several complex variables*, Stochastic Analysis, M. Pinsky, ed., Marcel-Dekker, (1984).
6. G. Kallianpur, D. Kannan and R.L. Karadikar, *Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces, and a Cameron-Martin formula*, to appear in Annals of prob.
7. H.H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture notes in Mathematics, No. 463, Springer-Verlag, Berlin (1975).

Sogang University

Seoul 121, Korea

and

Chonnam National University

Kwangju 500, Korea