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ON THE INITIAL CONVERGENCE STRUCTURE

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1. Convergence structure

D.C. Kent introduced the convergence structure and investigated some properties. With each convergence structure, there is an associated pretopology, the finest pretopology which is coarser than the convergence structure [2]. We introduces the notion of initial convergence structure and investigate some properties. For definition, not given here, the reader is asked to refer to [1], [2] and [3]. For a set X, $F(X)$ denotes the set of all filters on X and $P(X)$ the set of all subsets of X. For each $x\in X$, $\dot{x}$ is the principal ultrafilter containing $\{x\}$.

A convergence structure on X is a map q from $F(X)$ into $P(X)$ satisfying the following conditions:

(1) for each $x\in X$, $x\in q(\dot{x})$;

(2) for $\psi, \Phi\in F(X)$, if $\psi\subset\Phi$, then $q(\psi)\subset q(\Phi)$;

(3) if $x\in q(\psi)$, then $x\in q(\psi\cap\dot{x})$.

The pair (X, q) is called a convergence space. If $x\in q(\psi)$, we say that ψ q-converges to x. The filter $V_q(x)$ obtained by intersecting all filters which q-converge to x is called the q-neighborhood filter at x. If $V_q(x)$ q-converges to x for each $x\in X$, then q is called a pretopology, and (X, q) a pretopological space. Pretopology q is called a topology if for each $x\in X$, the filter $V_q(x)$ has a filterbase $B_q(x)\subset V_q$

(x) with the following property:

$y \in G(x) \in B_q(x)$ implies $G(x) \in B_q(y)$.

PROPOSITION 1.1. ([1]) *Let (X, q) be a convergence space. $\pi(q)$ defined by: ψ $\pi(q)$-converges to x if and only if $\psi \supset V_q(x)$ for each $x \in X$. Then $\pi(q)$ is the finest pretopology coarser than q.*

Let f be a map from a convergence space (X, q) onto a convergence space (Y, p). If $\psi \in F(X)$, then $f(\psi)$ will denote the filter on Y generated by $\{f(F) | F \in \psi\}$. f is continuous at a point $x \in X$ if for any filter ψ q-convergent to x, the filter $f(\psi)$ p-converges to $f(x)$. If f is continuous at every point $x \in X$, then f is said to be continuous.

PROPOSITION 1.2. ([3]) (1) *If f is continuous, then $f(V_q(x)) \supset V_p(f(x))$ for all $x \in X$.*

(2) *If p is a pretopology and $f(V_q(x)) \supset V_p(f(x))$ for all $x \in X$, then f is continuous.*

A convergence space (X, q) will be called almost pretopological if $q(\psi) = \pi(q)(\psi)$ for all ultrafilter ψ on X.

2. Initial convergence structure

Let X be a set, (Y_α, p_α) be a convergence space for each $\alpha \in \Lambda$, $f_\alpha : X \longrightarrow (Y_\alpha, p_\alpha)$ be a surjection. q is a map from $F(X)$ into $P(X)$ satisfying the following condition:

for any element $x \in X$, $\psi \in F(X)$, $x \in q(\psi)$ if and only if $f_\alpha(\psi)$ p_α-converges to $f_\alpha(x)$ for each $\alpha \in \Lambda$.

Then we obtain a convergence structure q on X that is said to be the initial convergence structure induced by the family $\{f_\alpha | \alpha \in \Lambda\}$.

The following facts can be easily verified.

PROPOSITION 2.1. *q is the coarest of all the convergence stucture on X which allow every f_α to be continuous for* *each* $\alpha \in \Lambda$.

PROPOSITION 2.2. *Map f from a convergence space (S, r) onto the convergence space (X, q) is continuous if and only if $f_\alpha \cdot f$ is continuous for each $\alpha \in \Lambda$.*

PROPOSITION 2.3. *Every convergence space of the family $\{(Y_\alpha, p_\alpha) | \alpha \in \Lambda\}$ is pretopological (topological) then q is a pretopology (topology).*

PROOF. If (Y_α, p_α) is pretopological for each $\alpha \in \Lambda$, then $V_{p_\alpha}(y_\alpha)$ p_α-converges to y_α for each $y_\alpha \in Y_\alpha$, $\alpha \in \Lambda$. Let $y_\alpha = f_\alpha(x_\alpha)$, $x_\alpha \in X$. By proposition 1.2,

$$V_{p_\alpha}(f_\alpha(x_\alpha)) \subset f_\alpha(V_q(x_\alpha)).$$

Thus $f_\alpha(V_q(x_\alpha))$ p_α-converges to y_α for each $\alpha \in \Lambda$. Therefore $V_q(x_\alpha)$ q-converges to x_α. The "topology" part can be proved similarly.

The product $X = \prod_{\alpha \in \Lambda} X_\alpha$ of a family $\{(X_\alpha, q_\alpha) | \alpha \in \Lambda\}$ of convergence spaces is the product of the underlying sets of the family endowed with the initial convergence structure q induced by the family of all the canonical projections $P_\alpha : X \longrightarrow X_\alpha$ for each $\alpha \in \Lambda$.

PROPOSITION 2.4. *For all points $x = (x_\alpha)_{\alpha \in \Lambda}$ in X*

(1) $V_q(x) \geqq \prod_{\alpha \in \Lambda} V_{q_\alpha}(x_\alpha)$

(2) $P_\alpha(V_q(x)) = V_{q_\alpha}(x_\alpha)$.

PROOF. (1) Let $G = \prod_{\alpha \in \Lambda} G_\alpha$ be any element of $\prod_{\alpha \in \Lambda} V_{q_\alpha}(x_\alpha)$, then there exists a $G'_\alpha \in V_{q_\alpha}(x_\alpha)$ for each $\alpha \in \Lambda$ such that $\prod_{\alpha \in \Lambda} G'_\alpha \subset G$. Since for each $\alpha \in \Lambda$, $G'_\alpha \subset G_\alpha$, $G_\alpha \in V_{q_\alpha}(x_\alpha)$, let ψ be any filter in X with $x_\alpha \in q_\alpha(P_\alpha(\psi))$ for each $\alpha \in \Lambda$. Since $P_\alpha(\psi)$ is a filter in X_α, $V_{q_\alpha}(x_\alpha) \subset P_\alpha(\psi)$. Thus $G_\alpha \in P_\alpha(\psi)$, that is, $G \in \psi$, $G \in V_q(x)$.

(2) Since $V_q(x) \geqq \prod_{\alpha\in\Lambda} V_{q_\alpha}(x_\alpha)$, $P_\alpha(V_q(x)) \geqq V_{q_\alpha}(x_\alpha)$, for each $\alpha\in\Lambda$. Let G_α be any element of $P_\alpha(V_q(x))$, then there exists a $G'_\alpha \in V_{q_\alpha}(x_\alpha)$ such that $G'_\alpha \subset G_\alpha$ for each $\alpha\in\Lambda$. Since $V_{q_\alpha}(x_\alpha)$ is a filter on X_α, thus $G_\alpha \in V_{q_\alpha}(x_\alpha)$. Therefore $P_\alpha(V_q(x)) \leqq V_{q_\alpha}(x_\alpha)$.

PROPOSITION 2.5. *(X_α, q_α) is a pretopological space for each $\alpha\in\Lambda$ if and only if (X, q) is a pretopological space.*

PROOF. If (X_α, q_α) is a pretopological space for each $\alpha\in\Lambda$, then by proposition 2.3, (X, q) is a pretopological space.

Conversely, let q be a pretopology, then for each $x=(x_\alpha)_{\alpha\in\Lambda}\in X$, $x\in q(V_q(x))$, $x_\alpha \in q_\alpha(P_\alpha(V_q(x)))$. Thus q_α is a pretopology for each $\alpha\in\Lambda$.

PROPOSITION 2.6. *Let $(X, s) = \prod_{\alpha\in\Lambda}(X_\alpha, \pi(q_\alpha))$. Then the following statements holds:*

(1) *$\pi(q)$ is finer than s.*

(2) *$\pi(q) \neq s$.*

(3) *$\pi(q) = s$, if and only if $V_q(x) = \prod_{\alpha\in\Lambda} V_{q_\alpha}(x_\alpha)$, where $x=(x_\alpha)_{\alpha\in\Lambda}\in X$.*

PROOF. (1) Let ψ be any filter in X, x be any element of $\pi(q)(\psi)$, then $V_q(x)\subset\psi$. Thus $V_{q_\alpha}(x_\alpha) = P_\alpha(V_q(x)) \subset P_\alpha(\psi)$ for each $\alpha\in\Lambda$. If $x_\alpha \in \pi(q_\alpha)(P_\alpha(\psi))$, then $x=(x_\alpha)_{\alpha\in\Lambda} \in s(\psi)$. Therefore $\pi(q)(\psi) \subset s(\psi)$, and by [1] $\pi(q) \geqq s$.

(2) Trivial.

(3) Let $V_q(x) = \prod_{\alpha\in\Lambda} V_{q_\alpha}(x_\alpha)$, then by (1), $\pi(q) \geqq s$. If $x=(x_\alpha)_{\alpha\in\Lambda} \in s(\psi)$, then $x_\alpha \in \pi(q_\alpha)(p_\alpha(\psi))$ for each $\alpha\in\Lambda$. Since $V_{q_\alpha}(x_\alpha) \subset P_\alpha(\psi)$, $V_q(x) = \prod_{\alpha\in\Lambda} V_{q_\alpha}(x_\alpha) \subset \psi = \prod_{\alpha\in\Lambda} P_\alpha(\psi)$ and $x\in\pi(q)(\psi)$. Thus $\pi(q) = s$.

Conversely, let $\pi(q) = s$. By proposition 2.4,

$$\prod_{\alpha\in\Lambda} V_{q_\alpha}(x_\alpha)\subset V_q(x),$$

and by proposition 1.1, $\pi(q)$ is the finest pretopology coarser than q, $V_q(x)\subset\prod_{\in\Lambda} V_{q_\alpha}(x_\alpha)$.

PROPOSITION 2.7. *If (X_α,q_α) is an almost pretopological for each $\alpha\in\Lambda$, then $\pi(q)=s$.*

PROOF. By proposition 2.6, $\pi(q)\geqq s$. Suppose $\pi(q)>s$. Then by proposition 2.6, there exists a point $x=(x_\alpha)_{\alpha\in\Lambda}$ in X such that $\prod_{\alpha\in\Lambda} V_{q_\alpha}(x_\alpha)\subset V_q(x)$. Thus there exists an ultrafilter ψ in X such that $\psi\supset\prod_{\alpha\in\Lambda} V_{q_\alpha}(x_\alpha)$, but $\psi\not\supset V_q(x)$, $P_\alpha(\psi)\supset V_{q_\alpha}(x_\alpha)$ for each $\alpha\in\Lambda$. Since $P_\alpha(\psi)$ is an ultrafilter ~~in X_α~~ and q_α is an ~~almost pretopology~~ for each $\alpha\in\Lambda$, $q_\alpha(P(\psi))=\pi(q_\alpha)(P_\alpha(\psi))$. And $P_\alpha(\psi)\supset V_{q_\alpha}(x_\alpha)$, thus $x_\alpha\in\pi(q_\alpha)(P_\alpha(\psi))=q_\alpha(P_\alpha(\psi))$. Therefore $x=(x_\alpha)_{\alpha\in\Lambda}\in q(\psi)$. This contradicts the assumption that $\psi\supset V_q(x)$.

PROPOSITION 2.8. *Let (X,q) and (Y_α,p_α) be any convergence spaces for each $\alpha\in\Lambda$, $f:X\longrightarrow\prod_{\alpha\in\Lambda} Y_\alpha=Y$ be given by the equation $f(x)=(f_\alpha(x))_{\alpha\in\Lambda}$, where $f_\alpha:X\longrightarrow Y_\alpha$ for each $\alpha\in\Lambda$. Let Y have the initial convergence structure p induced by the family of all canonical projections $P_\alpha:Y\longrightarrow Y_\alpha$ for each $\alpha\in\Lambda$. Then the function f is continuous if and only if each function f_α is continuous.*

PROOF. Let ψ be any filter in X that q-converges to x for each $x\in X$. If f is continuous, then $f(\psi)$ p-converges to $f(x)$. Thus $f_\alpha(x)\in p_\alpha(P_\alpha(f(\psi)))=p_\alpha(f_\alpha(\psi))$ for each $\alpha\in\Lambda$. Therefore f_α is continuous.

Conversely, suppose that each coordinate function f_α is continuous, ψ q-converges to x for each $x\in X$. Then $f_\alpha(\psi)$ p_α-converges to $f_\alpha(x)$, that is, $f_\alpha(x)\in p_\alpha(f_\alpha(\psi))=p_\alpha$

$(P_\alpha(f(\psi)))$, thus $f(x)=(f_\alpha(x))_{\alpha\in\Lambda}\in p(f(\psi))$. Therefore f is continuous.

PROPOSITION 2.9. *Let f be a function as in proposition* 2.8.

(1) *If f is continuous, then* $f(V_q(x))\supset\prod_{\alpha\in\Lambda} V_{p_\alpha}(f_\alpha(x))$ *for all x in X.*

(2) *If p_α is a pretopology for each $\alpha\in\Lambda$ and* $f(V_q(x))\supset\prod_{\alpha\in\Lambda} V_{p_\alpha}(f_\alpha(x))$ *for all $x\in X$, then f is continuous.*

PROOF. (1) Since f is continuous, f_α is contnuous for each $\alpha\in\Lambda$, $V_{p_\alpha}(f_\alpha(x))\subset f_\alpha(V_q(x))$. Thus

$$\prod_{\alpha\in\Lambda} V_{p_\alpha}(f_\alpha(x))\subset\prod_{\alpha\in} f_\alpha(V_q(x))=f(V_q(x)).$$

(2) Let ψ be any filter in X and $x\in q(\psi)$, then

$$V_q(x)\subset\psi,\ \prod_{\alpha\in\Lambda} V_{p_\alpha}(f_\alpha(x))\subset f(V_q(x))\subset f(\psi).$$

Since p_α is a pretopology, $f_\alpha(x)\in p_\alpha(V_{p_\alpha}(f_\alpha(x)))$ and

$$f_\alpha(x)\in p_\alpha(V_{p_\alpha}(f_\alpha(x)))\subset p_\alpha(P_\alpha\prod_{\alpha\in\Lambda}(V_{p_\alpha}(f_\alpha(x)))).$$

Thus

$$f(x)=(f_\alpha(x))_{\alpha\in\Lambda}\in p(\prod_{\alpha\in\Lambda} V_{p_\alpha}(f_\alpha(x)))\subset p(f(V_q(x)))\subset p(f(\psi)).$$

Therefore $f(\psi)$ p-converges to $f(x)$.

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