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SOME REMARKS ON THE DISC ALGEBRA

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1. Introduction

Let $L^\infty = L^\infty(T)$ denote the space of essentially bounded measurable complex-valued functions with respect to the normalized Lebesque measure $(1/2\pi)\, d\theta$ defined on the unit circle T in the complex plane. With the usual operations and the norm

$$\|g\|_\infty = \text{ess sup}|g(e^{it})|,$$

L^∞ is a commutative Banach algebra. We denote by $C=C(T)$ the commutative Banach algebra of continuous complex-valued functions on T.

We denote by H^∞ and A the Banach algebras of functions in L^∞ and C, respectively, whose Fourier coefficients corresponding to the negative integers vanish.

For a complex-valued Lebesgue integrable function f on T, let $f(re^{i\theta})$ denote the harmonic extension of f into the unit disc D by means of Poisson's formula, that is,

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t)\, dt.$$

For a commutative Banach algebra B with identity, we let $M(B)$ denote the set of complex homomorphisms of B. $M(B)$ is called the maximal ideal space or spectrum of B. It is well known[4, p. 137] that $M(B)$ is a compact Hausdorff space with respect to the weak star topology, and $\|m\| = m$

$\hat{f}(1)=1$ for each complex homomorphism m in $M(B)$.

In this paper, we investigate some properties of the disc algebra A, and consider the condition that the functional on A of evaluation at the origin has a unique Hahn-Banach extension to a closed subalgebra of L^∞ containing A.

2. Main results

Let B be a commutative Banach algebra with identity. Writing

$$\hat{f}(m)=m(f),\ f\varepsilon B,\ m\varepsilon M(B),$$

we have a homomorphism $f\longrightarrow\hat{f}$ from B into $C(M(B))$, the algebra of continuous complex-valued functions on $M(B)$. This homomorphism is called the Gelfand transform. We note that the Gelfand transform is norm decreasing:

$$||\hat{f}||=\sup_{m\varepsilon M(B)}|\hat{f}(m)||\leqq||f||.$$

DEFINITION 2.1. The Banach algebra B is called a uniform algebra if the Gelfand transform is an isometry, that is, if

$$||\hat{f}||=||f||,\ f\varepsilon B.$$

It is well known [6, p. 270] that the Gelfand transform is an isometry if and only if $||f^2||=||f||^2$ for all $f\varepsilon B$.

EXAMPLE 2.2. *Let $f\varepsilon L^\infty$. Then $|f^2|=|f|\ |f|\leqq||f||^2_\infty$ almost everywhere, and $|f|=\sqrt{|f^2|}\leqq\sqrt{||f^2||_\infty}$ almost everywhere; hence $||f^2||_\infty=||f||^2_\infty$, and so L^∞ is a uniform algebra.*

Suppose B is any algebra of continuous complex-valued functions on a compact Hausdorff space Y. If B has the uniform norm and if B is complete, then B is a uniform algebra. If B contains the constant functions and separates the points of Y, we say that B is a uniform algebraon Y.

When B is a uniform algebra, the range $\hat{B}$ of the Gelfand transform is a uniformly closed subalgebra of $C(M(B))$, and $\hat{B}$ is isometrically isomorphic to B. In that case we identify f with $\hat{f}$ and write

$$f(m)=m(f)=\hat{f}(m),\ f\varepsilon B,\ m\varepsilon M(B).$$

Clearly, $B=\hat{B}$ separates the points of $M(B)$ and contains the constant functions on $M(B)$. Thus, any uniform algebra B is a uniform algebra on its maximal ideal space $M(B)$.

DEFINITION 2.3. *The uniform algebra B on a compact Hausdorff space Y is said to a Dirichlet algebra if $Re\ B=\{Re\ f : f\varepsilon B\}$ is uniformly dense in $C_R(Y)$, the algebra of real-valued continuous functions on Y.*

PROPOSITION 2.4. *The disc algebra A is a Dirichlet algebra.*

PROOF. It is clear that A is a uniform algebra on T. If f is a function in $C_R(T)$, every Cesaro mean for f is a real-valued trigometric polynomial:

$$\sigma_n(x)=\sum_{k=-(n-1)}^{n-1}(1-\frac{|k|}{n})c_k e^{ikx},$$

$$c_{-k}=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{ikt}dt=\bar{c}_k.$$

Clearly, $\sigma^n(x)\ \varepsilon\ Re\ A$. Since $\{\sigma_n\}$ converges uniformly to f, it follows that $Re\ A$ is dense in $C_R(T)$.

PROPOSITION 2.5. *The algebra $\bigcup_{n=1}^{\infty} z^{-n}A$ is dense in C.*

PROOF. Let P be the set of all trigonometric polynomials. Then we have

$$P\subset\bigcup_{n=1}^{\infty} z^{-n}A\subset C.$$

Since P is a self-adjoint subalgebra of C which contains the constant functions and separates points, it follows from the Stone-Weierstrass theorem that the uniform closure of P is C. Hence C is the uniform closure of $\bigcup_{n=1}^{\infty} z^{-n}A$.

PROPOSITION 2.6. *There exists a closed subalgebra of L^∞ which contains A, is not contained in H^∞, and does not contain C.*

PROOF. Let K be a closed nowhere dense subset of T of positive Lebesgue measure, and let B be the closed subalgebra of L^∞ generated by A and χ_K. Clearly, $\chi_K \varepsilon B$. Suppose $\chi_K \varepsilon H^\infty$. Since K is a set of positive measure, it follows that $\|\chi_K\|_\infty = 1$. Let g be the analytic extension of χ_K into D. Since $\chi_K = 0$ on $T-K$, it follows that $g(z) \equiv 0$ in D. (See [3, p. 76] or [5, p. 266]); hence

$$1 = \|\chi_K\|_\infty = \|g\|_\infty = 0.$$

This is a contradiction. Hence, $\chi_K \notin H^\infty$, and so B is not contained in H^∞.

To prove that B does not contain C, we note that B is the norm closure of the set $\{\chi_K g + h : g, h \varepsilon A\}$. Since $T-K$ is dense in T, it follows that

$$(2.1)\quad \left\|\chi_K g + h - \frac{1}{z}\right\|_\infty \geqq \text{ess sup}\{|(\chi_K g)(e^{it}) + h(e^{it}) - e^{-it}| : e^{it} \varepsilon T-K\}$$
$$= \text{ess sup}\{|h(e^{it}) - e^{-it}| : e^{it} \varepsilon T-K\}$$
$$= \sup\{|h(e^{it}) - e^{-it}| : e^{it} \varepsilon T\}.$$

Without loss of generality, we may assume that $h(0) \neq 0$. Then

$$(2.2)\quad \left\|h - \frac{1}{z}\right\|_\infty \geqq \sup_\theta \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, [h(e^{it}) - e^{-it}]\, dt \right|$$
$$= \frac{1+r}{1-r} |h(0)| \qquad (0 \leqq r < 1).$$

Choose r such that $\frac{1-|h(0)|}{1+|h(0)|} \leqq r<1$. Then, by(2.1) and (2.2), $||\chi_K g+h-\frac{1}{z}||_\infty \geqq 1$. Hence $z^{-1} \notin B$, and so C is not contained in B.

If Y is a Banach space and Y_0 is a subspace of Y, the Hahn-Banach theorem asserts that any bounded linear functional m_0 on Y_0 has a bounded linear extension m to Y satisfying $||m||=||m_0||$. If there is only one such m, then m_0 is said to have a unique Hahn-Banach extension. A bounded linear functional m on a Banach algebra B is said to be positive if $Re\ m(f) \geqq 0$ for every f in B with $Re\ f \geqq 0$. We note that if m is a linear functional on a subalgebra of L^∞ and $||m||=m(1)=1$, then m is positive. (See [1, p. 81]).

THEOREM 2.7. *Let B be a closed subalgebra of L^∞ containing the disc algebra A, and let m_0 be the bounded linear functional on A defined by $m_0(g)=g(0)$. Then m_0 has a unique Hahn-Banach extension to B if and only if*

$$(2.3) \quad sup\{Re\ m_0(g): g \in A,\ Re\ g \leqq Re\ f\} = inf\{Re\ m_0(g): g \in A,\ Re\ g \geqq Re\ f\}$$

for all $f \in B$.

PROOF. Suppose that m_0 has a unique Hahn-Banach extension to B. To establish (2.3), consider the positive linear functional on $Re\ A$ defined by

$$Re\ g \longrightarrow Re\ m_0(g).$$

By the proof of the Hahn-Banach theorem, this functional has a positive extension whose value at $Re\ f$ is any number in the interval $[\alpha_f, \beta_f]$, where

$$\alpha_f = \sup\{Re\ m_0(g) - ||Re\ g - Re\ f|| : g \in A\},$$

$$\beta_f = \inf\{Re\ m_0(g) + ||Re\ g - Re\ f|| : g \in A\}.$$

Since $Re(g-||Re\ g-Re\ f||) \leqq Re\ f \leqq Re(g+||Re\ g-Re\ f||)$, it follows that

$$\alpha_f \leqq \sup\{Re\ m_0(g) : g \varepsilon A,\ Re\ g \leqq Re\ f\}$$
$$\leqq \inf\{Re\ m_0(g) : g \varepsilon\ A,\ Re\ g \geqq Re\ f\} \leqq \beta_f.$$

So when the Hahn-Banach extension is unique, we must have equality in (2.3).

Conversely, suppose that (2.3) holds for all $f \varepsilon B$. Suppose that m_1 and m_2 are Hahn-Banach extensions of m_0 to B. Since $||m_0|| = m_0(1) = 1$, it follows that $||m_1|| = m_1(1) = 1$ and $||m_2|| = m_2(1) = 1$: hence m_1 and m_2 are positive. Assume that $m_1(f) \neq m_2(f)$ for some member f of B. Without loss of generality, we may assume that $Re\ m_1(f) < Re\ m_2(f)$. Put

$$\gamma_f = \sup\{Re\ m_0(g) : g \varepsilon A,\ Re\ g \leqq Re\ f\},$$
$$\delta_f = \inf\{Re\ m_0(g) : g \varepsilon A,\ Re\ g \geqq Re\ f\}.$$

If g is a member of A and $Re\ g \leqq Re\ f$, then $Re(f-g) \geqq 0$. Since m_1 is positive, $Re\ m_1(f) \geqq Re\ m_1(g) = Re m_0(g)$; hence

$$\text{(2.4)} \qquad Re\ m_1(f) \geqq \gamma_f.$$

Suppose that g is a member of A and $Re\ g \geqq Re\ f$. Since m_2 is positive, it follows that

$$Re\ m_0(g) = Re\ m_2(g) \geqq Re\ m_2(f);$$

hence

$$\text{(2.5)} \qquad \delta_f \geqq Re\ m_2(f).$$

By (2.4) and (2.5), we have $\delta_f > \gamma_f$. This contradicts the hypothesis that $\delta_f = \gamma_f$ for all f in B.

For a commutative Banach algebra with identity, there is a one-to-one correspondence $m \longleftrightarrow M$ between a complex homomorphism m of the algebra onto the algebra of complex numbers and a maximal ideal M in the algebra. The correspondence is defined by $M = ker(m)$ [2, p. 92]. Since every maximal ideal in a Banach algebra B is the kernel of a complex homomorphism $m : B \longrightarrow C$, it follows that $\hat{f}(m)$ is nowhere zero on $M(B)$ if and only if $f \varepsilon B^{-1}$, where B^{-1} is the set of all invertible elements of B.

THEOREM 2.8. *Let B be a closed subalgebra of L^{∞} containing the disc algebra A. Assume that the linear functional $m_0(f)=f(0)$ has a unique Hahn-Banach extension from A to B. Then either $B \supset C$ or $B \subset H^{\infty}$.*

PROOF. Suppose that B does not contain C. Then B doet not contain the function z^{-1} because, by proposition 2.5, the uniformly closed algebra generated by z^{-1} and A is C. Therefore the ideal zB in B consisting of all functions zf with f in B is proper. Consequently, there exists a complex homomorphism m of B such that $zB \subset ker(m)$. Since $f(z)-f(0) \varepsilon zB$ for all $f \varepsilon A$, it follows that m is is a Hahn-Banach extension of m_0. We note that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})d\theta = f(0) \text{ for every } f \text{ in } A;$$

hence the integration with respect to $\frac{1}{2\pi}d\theta$ defines a Hahn-Banach extension of m_0. Since m_0 has a unique Hahn-Banach extension to B, it follows that

$$m(f)=\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})d\theta$$

for every f in B. If f is any funtion in B and n is any positive integer, then the function $z^n f$ is in zB and so is annihilated by m, that is,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} f(e^{i\theta})d\theta = 0.$$

This says that the algebra B is contained in the algebra H^{∞}.

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