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PSEUDO-TOPOLOGICAL COHERENCE FOR CONVERGENCE SPACES

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1. Introduction

A convergence function is a correspondence between the filters on a given set X and the subsets of X. This concept is defined to include types of convergence which are more general than those defined by specifying a topology on X; the structures of convergence criteria are limitierung (H. R. Fischer), pseudo-topology (Choquet), and pretopology, but these structures may be regarded as special cases of convergence functions.

D. C. Kent and G. D. Richardson introduced the properties of convergence spaces, and applied such concepts as products of convergence spaces, pretopological coherence, and topological coherence. In this paper, we shall define pseudo-topological coherence and almost pseudo-topological spaces, and investigate their relations and properties.

2. Preliminaries

Our notation and terminology will concide with that of [1], [2], and [3]: however, a brief review of basic terms will be given in here.

Throughout this paper, we use the abbreviation "u. f." for ultrafilter, the contraction "iff" for "if and only if", and "$\mathscr{F}\xrightarrow{\mathscr{G}}x$" for "filter $\mathscr{F}$ $\mathscr{G}$-converges to x".

A convergence structure $\mathscr{q}$ on a set X is a mapping from the set $F(X)$ of all filters on X into the set $P(X)$ of all subsets of X which satisfies the following conditions:

(1) $\mathscr{F}, \mathscr{G}$ in $F(X)$ and $\mathscr{F} \subset \mathscr{G}$ implies $\mathscr{q}(\mathscr{F}) \subset \mathscr{q}(\mathscr{G})$;

(2) $x \in \mathscr{q}(\dot{x})$ for all x in X, where $\dot{x}$ denotes the u. f. generated by $\{x\}$;

(3) $x \in \mathscr{q}(\mathscr{F})$ implies $x \in \mathscr{q}(\mathscr{F} \cap \dot{x})$, where $\mathscr{F} \cap x$ is generated by intersection members of filters $\mathscr{F}$ and $\dot{x}$.

The pair $(X, \mathscr{q})$ is called a convergence space and "$x \in \mathscr{q}(\mathscr{F})$" is interpreted "$\mathscr{F}$ $\mathscr{q}$-converges to x".

Given convergence spaces $(X, \mathscr{q})$ and (Y, p), let R be the set product $X \times Y$ and let P_X be the canonical projection of R onto X, and P_Y the projection of R onto Y. The product convergence structure r on $X \times Y$ is defined by specifying that $\mathscr{K} \xrightarrow{r} (x, y)$ iff $P_X(\mathscr{K}) \xrightarrow{\mathscr{q}} x$ and $P_Y(\mathscr{K}) \xrightarrow{p} y$ where $P_X(\mathscr{K})$ is a filter generated by projection members for each K in a filter $\mathscr{K}$ on R, and the pair (R, r) is called the product space of $(X, \mathscr{q})$ and (Y, p), see [1].

In [6], if $\mathscr{F}$ and $\mathscr{G}$ are filters on X, we say $\mathscr{F}$ finer than $\mathscr{G}$ (or $\mathscr{G}$ is coarser than $\mathscr{F}$) iff $\mathscr{F} \supset \mathscr{G}$, we denote $\mathscr{F} \geq \mathscr{G}$. A filter $\mathscr{F}$ on X is fixed iff $\cap \mathscr{F} \neq \phi$ and free iff $\cap \mathscr{F} = \phi$. A filter $\mathscr{F}$ is an u. f. iff there is no strictly finer filter $\mathscr{G}$ than $\mathscr{F}$, thus the ultrafilters are the maximal filters.

In [1], given filters $\mathscr{F}$ on X and $\mathscr{G}$ on Y, let $\mathscr{F} \times \mathscr{G}$ denote the filter on $R (= X \times Y)$ generated by sets of the *form* $F \times G$, *for* F *in* $\mathscr{F}$ *and* G *in* $\mathscr{G}$.

3. Pseudo-topological coherence

In this section, we shall extend some properties of [1] to pseudo-topological spaces of finite convergence spaces.

A pseudo-topological space([4]) is a convergence space with the property that $\mathscr{F} \longrightarrow x$ whenever each u.f. finer than $\mathscr{F}$ converges to x.

In [3], for each convergence structure q on a set X, the following related convergence structure is defined by D. C. Kent: $\mathscr{F} \xrightarrow{\rho(q)} x$ iff $\mathscr{F}' \xrightarrow{q} x$ for each u.f. $\mathscr{F}'$ finer than $\mathscr{F}$; q is finer than $\rho(q)$ ($q \geq \rho(q)$) in the sense that $\mathscr{F}$ $\rho(q)$-converges to x whenver $\mathscr{F}$ q-converges to x, indeed, $\rho(q)$ is the finest pseudotopology coarser than q.

LEMMA 3.1. *If $\mathscr{K}$ is a filter on $R(=X \times Y)$, then for each u.f. $\mathscr{F}'$ finer than $P_X(\mathscr{K})$ on X, there exists an u.f. $\mathscr{G}'$ on Y such that $\mathscr{G}' \geq P_Y(\mathscr{K})$ and $\mathscr{F}' \times \mathscr{G}' \geq \mathscr{K}$.*

PROOF. Let $\mathscr{K}$ be a filter on $R(=X \times Y)$. Given an u.f. $\mathscr{F}'$ finer than $P_X(\mathscr{K})$ on X, let $\cap P_X(\mathscr{K}) = F$, then there exists x in F such that $\dot{x} = \mathscr{F}'$([6]) because $P_X(\mathscr{K})$ is fixed filter. Since $F \in \mathscr{F}'$, $\mathscr{F}' \geq P_X(\mathscr{K})$. Since $P_X^{-1}(\{x\}) \cap (\cap \mathscr{K}) \neq \phi$, we can obtain some nonempty subset G of $\cap P_Y(\mathscr{K})$ such that $\{x\} \times G \subset \cap \mathscr{K}$. Let $\mathscr{G}'$ be an u.f. containing G, then $\mathscr{G}' \geq P_Y(\mathscr{K})$ and $\mathscr{F}' \times \mathscr{G}' \geq \mathscr{K}$ because $\{x\} \times G \in \mathscr{F}' \times \mathscr{G}'$ and $\{x\} \times G \subset \cap \mathscr{K}$.

THEOREM 3.2. *If (X, q) and (Y, p) are pseudo-topological spaces, then $(X, q) \times (Y, p)$ is a pseudo-topological space.*

PROOF. Suppose that (X, q) and (Y, p) are pseudo-topological spaces, let (R, r) be the product convergence space of (X, q) and (Y, p). Given a filter $\mathscr{K}$ on R, let $\mathscr{K}' \xrightarrow{r} (x, y)$ for all u.f. $\mathscr{K}'$ finer than $\mathscr{K}$. If $\mathscr{F}'$ is an u.f. on X finer than $P_X(\mathscr{K})$, then, by Lemma 3.1, there exists the filter $\mathscr{F}' \times \mathscr{G}'$ on R such that $\mathscr{F}' \times \mathscr{G}' \geq \mathscr{K}$ where $\mathscr{G}'$ is the u.f. finer than $P_Y(\mathscr{K})$ on Y. Then there exists an u.f. $\mathscr{H}'$ such that $\mathscr{H}' \geq \mathscr{F}' \times \mathscr{G}' \geq \mathscr{K}$, so that $P_X(\mathscr{H}') \geq P_X(\mathscr{F}' \times \mathscr{G}') = \mathscr{F}' \geq$

$P_X(\mathscr{K})$, hence $\mathscr{F}'=P_X(\mathscr{K}')\xrightarrow{q}x$ by assumption. Similialy $\mathscr{G}'\xrightarrow{p}y$ for each u. f. $\mathscr{G}'$ finer than $P_Y(\mathscr{K})$ on Y. Since (X, q) and (Y, p) are pseudo-topological, $P_X(\mathscr{K})\xrightarrow{q}x$ and $P_Y(\mathscr{K})\xrightarrow{p}y$, and so $\mathscr{K}\xrightarrow{r}(x,y)$. Thus $(X,q)\times(Y,p)$ is a pseudo-topological space.

For any convergence space X, let ρX be the convergence space defined on the same underlying set as follows: $\mathscr{F}\longrightarrow x$ in ρX iff $\mathscr{G}\longrightarrow x$ in X for each u. f. $\mathscr{G}$ finer than $\mathscr{F}$. The space ρX is the finest pseudo-topological space coarser than X([4]), and it is called the pseudo-topological modification of X. Note that X and ρX have the same u. f. convergence ([4]).

DEFINITION 3.3. *A convergence space* (X, q) *is almost pseudo-topological iff* $q(\mathscr{F})=\rho(q)(\mathscr{F})$ *for all u.f.* $\mathscr{F}$ *on* X, *i.e., if* X *and* ρX *have the same u.f. convergence.*

PROPOSITION 3.4. $\rho X\times\rho Y\leq\rho(X\times Y)\leq X\times Y$ *for two convergence spaces* X *and* Y.

PROOF. By definition of pseudo-topological modification, $\rho(X\times Y)\leq X\times Y$. We shall show that $\rho X\times\rho Y\leq\rho(X\times Y)$. If $\mathscr{K}\to(x,y)$ in $\rho(X\times Y)$, then $\mathscr{K}'\to(x,y)$ in $X\times Y$ for all u. f. $\mathscr{K}'$ finer than $\mathscr{K}$. That is, $P_X(\mathscr{K}')$ converges to x in X ([1]), and so $P_X(\mathscr{K}')\to x$ in ρX by $\rho X\leq X$. Similiarly, $P_Y(\mathscr{K}')\to y$ in ρY. Thus, $\mathscr{K}'\to(x,y)$ in $\rho X\times\rho Y$ for all u. f. $\mathscr{K}'$ finer than $\mathscr{K}$. Since $\rho X\times\rho Y$ is a pseudo-topological space by Theorem 3.2, $\mathscr{K}\to(x,y)$ in $\rho X\times\rho Y$. Hence, $\rho X\times\rho Y\leq\rho(X\times Y)\leq X\times Y$.

THEOREM 3.5. *If* X *and* Y *are almost pseudo-topological, then* $X\times Y$ *is almost pseudo-topological.*

PROOF. Suppose that X and Y are almost pseudo-topological. If $\mathscr{K}'$ is an u.f. on $X\times Y$ such that $\mathscr{K}'\to(x,y)$ in $\rho(X\times Y)$, then $\mathscr{K}'\to(x,y)$ in $\rho X\times\rho Y$ by Proposition 3.4, and so $P_X(\mathscr{K}')\to x$ in ρX and $P_X(\mathscr{K}')$ is an u.f. on X. Since X is almost pseudo-topological, $P_X(\mathscr{K}')\to x$ in X. Similiarly, we can prove that $P_Y(\mathscr{K}')\to y$ in Y. Thus $\mathscr{K}'\to(x,y)$ in $X\times Y$([1]). The other hand, if $\mathscr{K}'$ is an u.f. on $X\times Y$ such that $\mathscr{K}'\to(x,y)$ in $X\times Y$, then $\mathscr{K}'\to(x,y)$ in $\rho(X\times Y)$ by Proposition 3.4. Therefore, $X\times Y$ is almost pseudo-topological.

DEFINITION 3.6. *A pair of convergence spaces X, Y is said to be pseudo-topologically coherent if $\rho(X\times Y)=\rho X\times\rho Y$.*

THEOREM 3.7. *X and Y are almost pseudo-topological iff the pair X, Y is pseudo-topologically coherent.*

PROOF. If $\rho X\times\rho Y<\rho(X\times Y)$, there exists a filter $\mathscr{K}$ on $X\times Y$ such that $\mathscr{K}\to(x,y)$ in $\rho X\times\rho Y$ and does not in $\rho(X\times Y)$. By definition of $\rho(X\times Y)$, there exists an u.f. $\mathscr{K}'$ such that $\mathscr{K}'$ does not converge to (x,y) in $X\times Y$ and $\mathscr{K}'\geq\mathscr{K}$. But $\mathscr{K}'\to(x,y)$ in $\rho X\times\rho Y$ because $\mathscr{K}\to(x,y)$ in $\rho X\times\rho Y$ and $\mathscr{K}'\geq\mathscr{K}$. Thus $P_X(\mathscr{K}')\to x$ in ρX and $P_Y(\mathscr{K}')\to y$ in ρY, so that $P_X(\mathscr{K}')\to x$ in X and $P_Y(\mathscr{K}')\to y$ in Y by Definition 3.3, hence $\mathscr{K}'\to(x,y)$ in $X\times Y$. This contradicts the fact that an u.f. $\mathscr{K}'$ does not converge to (x,y) in $X\times Y$, thus the pair X, Y is pseudo-topologically coherent.

Conversely, suppose that the pair X, $Y$is pseudo-topologically coherent, and let $\mathscr{F}'$ and $\mathscr{G}'$ be arbitrary u.f.'s on X and Y respectively. If $\mathscr{F}'\to x$ in X and $\mathscr{G}'\to y$ in Y, since $\rho X\leq X$ and $\rho Y\leq Y$, $\mathscr{F}'\to x$ in ρX and $\mathscr{G}'\to y$ in ρY. The other hand, if $\mathscr{F}'\to x$ in ρX and $\mathscr{G}'\to y$ in ρY, then $\mathscr{F}'\times\mathscr{G}'\to(x,y)$

in $\rho X\times\rho Y$. Since $\rho(X\times Y)=\rho X\times\rho Y$, $\mathscr{F}'\times\mathscr{G}'\to(x,y)$ in $\rho(X\times Y)$, and so $\mathscr{H}'\to(x,y)$ in $X\times Y$ for all u. f. $\mathscr{H}'$ finer than $\mathscr{F}'\times\mathscr{G}'$, thus $P_X(\mathscr{H}')\to x$ in X and $P_Y(\mathscr{H}')\to y$ in Y. Since $P_X(\mathscr{H}')$ and $\mathscr{F}'$ are the u. f. on X and $P_X(\mathscr{H}')\geq P_X(\mathscr{F}'\times\mathscr{G}')=\mathscr{F}'$, $P_X(\mathscr{H}')=\mathscr{F}'\to x$ in X, similiarly $\mathscr{G}'\to y$ in Y. Thus X and ρX have the same u. f. convergence, hence X and Y are almost pseudo-topological.

From Theorem 3. 5 and 3. 7, we can obtain the following corollary.

COROLLARY 3. 8. *If the pair X, Y is pseudo-topologically coherent, then $X\times Y$ is almost pseudo-topological.*

THEOREM 3. 9. (a) *If (X,q) and (Y,p) form a pseudo-topologically coherent pair, $\rho(q_1)=\rho(q)$, $\rho(p_1)=\rho(p)$, $q_1\leq q$, and $p_1\leq p$, then (X,q_1) and (Y,p_1) also form a pseudo-topologically coherent pair.*

(b) *If (X,q) and (Y,p) do not form a pseudo-topologically coherent pair, $\rho(q_1)=\rho(q)$, $\rho(p_1)=\rho(p)$, $q_1\geq q$, and $p_1\geq p$, then the pair (X,q_1), (Y,p_1) also fails to be pseudotopologically coherent.*

PROOF. (a) $\rho(q_1\times p_1)\leq\rho(q\times p)=\rho(q)\times\rho(p)=\rho(q_1)\times\rho(p_1)$ and by Proposition 3. 4, $\rho(q_1\times p_1)\geq\rho(q_1)\times\rho(p_1)$. Thus $\rho(q_1\times p_1)=\rho(q_1)\times\rho(p_1)$, that is, (X,q_1) and (Y,p_1) is a pseudo-topologically coherent pair.

(b) By assumption, $\rho(q_1)\times\rho(p_1)=\rho(q)\times\rho(p)<\rho(q\times p)\leq\rho(q_1\times p_1)$, that is, $\rho(q_1)\times\rho(p_1)\neq\rho(q_1\times p_1)$. Thus the pair (X,q_1), (Y,p_1) fails to be pseudo-topologically coherent.

THEOREM 3. 10. *If X and Y are convergence spaces such that $X\times\rho Y\geq\rho(X\times Y)$, then the pair X, Y is pseudo-topologically coherent iff the pair X, ρY is pseudo-topologically coherent.*

PROOF. Suppose that $X\times\rho Y\geq\rho(X\times Y)$. If the pair X,Y is pseudo-topologically coherent, then $\rho(X\times\rho Y)\leq\rho(X\times Y)$ $=\rho X\times\rho Y=\rho X\times\rho(\rho Y)$ and $\rho(X\times\rho Y)\geq\rho(\rho(X\times Y))=\rho(X\times Y)=\rho X\times\rho Y=\rho X\times\rho(\rho Y)$. Thus $\rho(X\times\rho Y)=\rho X\times\rho(\rho Y)$, so that the pair X, ρY is pseudo-topologically coherent.

Conversely, if the pair X, ρY is pseudo-topologically coherent, since $\rho(X\times Y)\leq X\times\rho Y$, $\rho(X\times Y)=\rho(\rho(X\times Y))\leq\rho(X\times\rho Y)=\rho X\times\rho Y$, that is, $\rho(X\times Y)=\rho X\times\rho Y$. Thus the pair X,Y is pseudo-topplogically coherent.

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