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NEARNESS AND QUASI-PROXIMITY INDUCED BY QUASI-UNIFORMITY

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1. Introduction

A generalization of the entourage definition yields quasi-uniform structures. The paper [3] characterized those quasi-uniform spaces for which the collection of quasi-uniform covers is a nearness structure with the same topological closure operator. The characterizing property is called locally symmetric.

Using this property, this paper shows that quasi-proximity spaces induced by a quasi-uniformity have the same property and a quasi-proximity is a nearness structure with the same topological closure operator.

2. Preliminaries

Let X be a set, then $P^n(X)$ will denote the power set of $P^{n-1}(X)$ for each natural number n and $P^0(X)=X$. Let ξ be a subset of $P^2(X)$ and $\mathscr{A}$ and $\mathscr{B}$ subsets of $P(X)$. Let A and B be subsets of X. Then the following notation is used:

(a) $\mathscr{A}$ is near means $\mathscr{A}\in\xi$.

(b) $cl_\xi\ A=\{x\in X:\{\{x\},\ A\}\in\xi\}$.

(c) $\mathscr{A}\vee\mathscr{B}=\{A\cup B:\ A\in\mathscr{A},\ B\in\mathscr{B}\}$.

DEFINITION 2.1 [1], [2]. *Let X be a set and $\xi\subset P^2(X)$. Then (X,ξ) is called a nearness space-shortly N-space provided:*

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(N1) $\cap\mathscr{A}\neq\phi$ *implies* $\mathscr{A}\in\xi$.

(N2) *If* $\mathscr{A}\in\xi$ *and for each* $B\in\mathscr{B}$ *there exists* $A\in\mathscr{A}$ *with* $A\subset cl_\xi B$, *then* $\mathscr{B}\in\xi$.

(N3) *If* $\mathscr{A}\notin\xi$, *and* $\mathscr{B}\notin\xi$ *then* $\mathscr{A}\vee\mathscr{B}\notin\xi$.

(N4) $\phi\in\mathscr{A}$ *implies* $\mathscr{A}\notin\xi$.

Given a N-space(X,ξ), the operator cl_ξ is a closure operator on X. Hence there exists a topology associated with each N-space in a natural way. This topology is symmetric denoted by $t(\xi)$. (Recall that a topology is symmetric provided $x\in\{\bar{y}\}$ implies $y\in\{\bar{x}\}$.) Conversely, given any symmetric topological space (X,t) there exists a compatible nearness structure [4]- shortly N-structure, ξ_t given by $\xi_t=\{\mathscr{A}\subset P(X):\cap\overline{\mathscr{A}}\neq\phi\}$. To say that a N-structure ξ is compatible with a topology t on a set X means that $t=t(\xi)$.

DEFINITION 2.2. *If* (X,ξ) *is a* N*-space then the relation* δ *on* $P(X)$ *defined by* $\{A,B\}\in\xi$ *iff* $A\delta B$ *and* $\{A,B\}\notin\xi$ *iff* $A\not\delta B$. *A relation* δ *in* $P(X)$ *is a quasi-proximity for a set* X *if it satisfies the following conditions*:

(Pa) $X\not\delta\phi$ *and* $\phi\not\delta X$.

(Pb) $C\delta A\cup B$ *iff* $C\delta A$ *or* $C\delta B$.
$A\cup B\delta C$ *iff* $A\delta C$ *or* $B\delta C$.

(Pc) $\{x\}\delta\{x\}$ *for each* $x\in X$.

(Pd) *If* $A\not\delta B$, *there exists* $C\subset X$ *such that* $A\not\delta C$ *and* $X\text{-}C\not\delta B$.

The pair (X,δ) *is called a quasi-proximity space. If* δ *is a quasi-proximity on* X *then so is* δ^{-1}. *A quasi-proximity* δ *is a proximity if* $\delta=\delta^{-1}$.

Let A *and* B *be subsets of a quasi-proximity space* (X,δ). *If* $A\delta B$, *then* A *is said to be near* B *and if* $A\not\delta B$, *then* A *is said to be far from* B.

DEFINITION 2.3. *A set B is said to be a δ-neighborhood of a set A if $A \not\delta X-B$.*

PROPOSITION 2.4. *If (X, δ) is a quasi-proximity space, the function $cl_\delta : P(X) \to P(X)$ defined by $cl_\delta(A) = \{x : \{x\} \delta A\}$ is a closure operator on X.*

If (X, δ) is a quasi-proximity space, then the topology induced by δ(or simply the topology $T(\delta)$ of δ) is the topology generated by the closure operator defined in proposition 2.4. A quasi-proximity δ is compatible with a topology T provided δ induces T. Of $x \in X$, the $T(\delta)$-neighborhoods of x are precisely the δ-neighborhoods of x.

PROPOSITION 2.5. *Let (X, δ) be a quasi-proximity space. If $\{x\} \not\delta A$, there is a $T(\delta)$-neighborhood G of x such that $G \not\delta A$.*

PROOF. Since $\{x\} \not\delta A$, there is a set C such that $\{x\} \not\delta C$ and X-$C \not\delta A$. Set $G = \text{int}(X-C)$. As noted above, $x \in G$ and $G \not\delta A$.

DEFINITION 2.6. *Let X be a set. A quasi-uniform structure $\mathcal{U}$ on X is a filter on $X \times X$ satisfying:*

(U1) $\{(x, x) : x \in X\} \subset U$ for each $U \in \mathcal{U}$.

(U2) For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The topology induced by $\mathcal{U}$ is the topology denoted by $T(\mathcal{U})$.

PROPOSITION 2.7. *Let $\mathcal{U}$ be a quasi-uniformity on X and let $\delta_\mathcal{U}$ denote the relation in $P(X)$ defined by $A \delta_\mathcal{U} B$ provided that for each $U \in \mathcal{U}$, $A \times B \cap U \neq \phi$. Then $\delta_\mathcal{U}$ is a quasi-proximity on X, and $T(\mathcal{U}) = T(\delta_\mathcal{U})$.*

DEFINITION 2.8. *The quasi-proximity induced by $\mathcal{U}$ is the quasi-proximity of proposition 2.7, defined by $A \delta_\mathcal{U} B$.*

A quasi-uniformity $\mathscr{U}$ *is said to be compatible with* δ *if* $\delta_{\mathscr{U}}=\delta$.

DEFINITION 2.9. *A quasi-uniform space*$(X,\mathscr{U})$ *is totally bounded provided that for each (subbasic) entourage* U *there is a finite cover* $\mathscr{A}$ *of* X *such that for each* $A\in\mathscr{A}$, $A\times A\subset U$.

Let (X,δ) be a quasi-proximity space. We proceed to show that there is a unique totally bounded quasi-uniformity compatible with δ and that this quasi-uniformity is the coarsest quasi-uniformity compatible with δ. The result is fundamental. It establishes a one-to-one correspondence between the quasi-proximities and the totally bounded quasi-uniformities that are compatible with a given topology.

Let X be a set and $(A,B)\in P(X)\times P(X)$. And let $T(A,B)$ denote $X\times X-A\times B$.

THEOREM 2.10 [5]. *Let* (X,δ) *be a quasi-proximity space. The collection* S *of all sets of the form* $T(A,B)$, *where* $A\bar{\delta}B$, *is a subbase for a totally bounded quasi-uniformity,* $\mathscr{U}_\delta$, *which is compatible with* δ.

3. Results

Although the difference between proximity spaces and arbitrary quasi-proximity spaces is only a matter of assuming an axiom of symmetry, the difference between these two classes of spaces is considerable. We investigate some approximations of symmetry, which serve to narrow the gap between the quasi-proximities and proximities. Obviousely, N-structures and quasi-proximity structures agree if they are proximity structures.

DEFINITION 3.1. *A quasi-proximity* δ *on a set* X *is point* *symmtric provided that* $\{x\}\,\delta A$ *whenever* $A\delta\{x\}$.

PROPOSITION 3.2. *Let* $(X, \mathcal{U})$ *be a (totally bounded) quasi-uniform space then the following statements are equivalent*:

(TBa) $(X, \delta_{\mathcal{U}})$ *is point symmetric.*

(TBb) *For each* $U\in\mathcal{U}$ *and* $x\in X$, *there exists a symmetric* $V\in\mathcal{U}$ *such that* $V(x)\subset U(x)$.

PROOF. Prove only that (TBa) implies (TBb). Since the remaining implications are apparent. Let $U\in\mathcal{U}$ and let $x\in X$. Then $\{x\}\not\delta_{\mathcal{U}}\ X-U(x)$ so that $X-U(x)\not\delta_{\mathcal{U}}\{x\}$. Set $V=T(\{x\},\ X-U(x)\}\cap T(X-U(x),\ \{x\})$. By theorem 2.10, $V\in\mathcal{U}$. Furthermore V is symmetric and $V(x)\subset U(x)$.

REMARK 3.3. From the preceding proposition that a topological space that admits a point symmetric quasi-proximity is a symmetric space. Moreover if (X, t) is a symmetic space, then δ_p is a point symmetric quasi-proximity.

DEFINITION 3.4. *A quasi-proximity* δ *and a set* X *is locally symmetric provided that* $\{x\}\delta A$ *whenever* $A\delta G$ *for each* $T(\delta)$*-neighborhood* G *of* x.

REMARK 3.5. By proposition 2.5, every proximity is locally symmetric, and both point symmetric and locally symmetric are hereditry properties.

PROPOSITION 3.6. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space. The following statements are equivalent*:

(a) $(X, \delta_{\mathcal{U}})$ *is locally symmetric.*

(b) *For each* $U\in\mathcal{U}$ *and* $x\in X$, *there exists a symmetric* $V\in\mathcal{U}$ *such that* $V^{-1}(V(x))\subset U(x)$.

(c) *For each* $U\in\mathcal{U}$ *and* $x\in X$, *there exists a* $V\in\mathcal{U}$ *such that* $V^2(x)\subset U(x)$.

(d) *For each* $x\in X$, $\{U^{-1}(U(x)):U\in\mathcal{U}\}$ *is a base for* *the* $T(\mathscr{U})$*-neighborhood filter* of x

DEFINITION 3.7. *Let* $(X,\mathscr{U})$ *be a quasi-unifoim space.* $(X,\mathscr{U})$ *is called locally symmetric provided it satisfies any of the conditions of the previous propositions.*

COROLLARY 3.8. $(X,\delta_{\mathscr{U}})$ *is locally symmetric iff* $(X,\mathscr{U})$ *is locally symmetric.*

PROOF. By proposition 3.6 and definition 3.7, it is obvious.

REMARK 3.9. If $(X,\delta_{\mathscr{U}})$ is locally symmetric, then $T(\delta_{\mathscr{U}})$ is symmetric.

DEFINITION 3.10. *Let* $(X,\delta_{\mathscr{U}})$ *be a quasi-proximity space. Set* $\xi(\delta_{\mathscr{U}})=\{\mathscr{A}\subset P(X):\bigcap\{cl_{\delta_{\mathscr{U}}}A:A\in\mathscr{A}\}\neq\phi\}$.

We know that the quasi-proximity spaces is coincide with the N-space for the closure operator.

THEOREM 3.11. *Let* $(X,\mathscr{U})$ *be a quasi-uniform space. The following statements are equivalent*:

(a) $(X,\delta_{\mathscr{U}})$ *is locally symmetric.*

(*b*) $(X,\xi(\delta_{\mathscr{U}}))$ *is a N-space and* $cl_{\delta_{\mathscr{U}}}(A)=cl_{\xi(\delta_{\mathscr{U}})}(A)$ *for each* $A\subset X$.

PROOF. Axioms($N1$), ($N2$), and ($N4$) are easy to verify. Enough to show that ($N3$). Now suppose $\mathscr{A}\in\xi(\delta_{\mathscr{U}})$ and for each $B\in\mathscr{B}$ there exists $A\in\mathscr{A}$ such that $A\subset cl_{\xi(\delta_{\mathscr{U}})}(B)$. Since $(X,\delta_{\mathscr{U}})$ is locally symmetric, $\bigcap cl_{\delta_{\mathscr{U}}} A\subset\bigcap cl_{\xi(\delta_{\mathscr{U}})} B=\bigcap cl_{\delta_{\mathscr{U}}}B$, where $B\in\mathscr{B}$. Thus $\mathscr{B}\in\xi(\delta_{\mathscr{U}})$. Therefore $\xi(\delta_{\mathscr{U}})$ is a N-structure, converse is obvious follows from the property of closure operator.

We now have a bridge from locally symmetric quasi- proximity spaces to N-spaces follows from this paper and [3]. $\xi(\delta_{\mathscr{U}})$ can be thought as the underlying N-structure in a natural way.

References

[1] H. Herrlich, A concept of nearness, Gen. Topology Appl., 5 (1974) 191-212.

[2] H. Herrlich, Topological structures, Math. Centre Tracts 52 (1974) 59-122.

[3] J. W. Carlson, Nearness and quasi-uniform structures, Gen. Topology Appl., 15 (1983) 19-28.

[4] W. N Hunsaker and P. L. Sharma, Nearness structures compatible with a topological space, Arch. Math (Basel) 25 (1974) 172-177.

[5] P. Fletcher and W. F. Lindgren, Quasi-uniform spaces, Lecture Notes in pure and Appl. Math. 77 (1981) 10-37.

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