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ASYMPTOTIC BEHAVIOUR OF NONLINEAR SEMIGROUPS IN BANACH SPACES

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1. Introduction.

Throughout this paper E denotes a real Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and S the nonlinear semigroup generated by $-A$. The main purpose of the present paper is to show that the weak and strong convergence of $J_t x/t$ and $S(t)x/t$ as $t \to \infty$ and the properties of the range of A. We also derive the weak and strong convergence of $(x - J_t x)/t$ and $(x - S(t)x)/t$ as $t \to 0+$. In this direction were established by Crandall [1, p.166] and Pazy [5] in Hilbert space. For recent development in Banach space see the papers by Kohlberg and Neyman [3, 4]

2. Preliminaries

Let E be a real Banach Space, and let I denote the identity operator. Then an operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ is said to be an acccretive if $|x_1 - x_2| \leqq |x_1 - x_2 + r(y_1 - y_2)|$ for all $y \in Ax$, $i=1,2$, and $r>0$. If A is an accretive, we can define, for each positive r, the resolvent J_r: $R(1+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$ and the Yosida approximation A_r: $R(I+rA) \to R(A)$ by $A_r = (I - J_r)/r$.

We know that $A_r x \in AJ_r x$ for every $x \in R(I+rA)$ and

$|A_r x| \leqq \|Ax\|$ for every $x \in D(A) \cap R(I+rA)$, where $\|Ax\| = \inf\{|y| : y \in Ax\}$,

We denote the closure of a subset D of E by $cl(D)$ and its closed convex hull by $\overline{Co}D$. We shall say that A satisfies the range condition if $R(1+rA) \supset cl(D(A))$ for all $r>0$. In this case, $-A$ generates a nonexpansive nonlinear semigroups $S: [0, \infty) \times cl(D(A)) \to cl(D(A))$ by $S(t)x = \lim_{n \to \infty} (I+(t/n)A)^{-n}x$.

We also prove that that $A^{-1}0 = F(J_r)$ for each $r>0$, where $F(J_r)$ is the set of fixed points of J_r.

Let S be a set and let $m(S)$ be the Banach space of all bounded real valued functions on S with the supremum norm, An element $\mu \in m(S)^*$ (the dual space of $m(S)$) is called a mean on S if $|\mu| = \mu(1) = 1$. Let μ be a mean on S and $f \in m(S)$. Then we denote by $\mu(f)$ the value of μ at the function f. According to the time and circumstance, we write by $\mu_t f(t)$ the value of $\mu(f)$. We know that $\mu \in m(S)^*$ is a mean on S if and only if

$$\inf\{f(s) : s \in S\} \leqq \mu(f) \leqq \sup\{f(s) : s \in S\}$$

for every $f \in m(S)$. Let S be an abstract semigroup. Then, for each $s \in S$ and $f \in m(S)$, we can define elements f_s and f^s in $m(S)$ given by $f_s(t) = f(st)$ and $f^s(t) = f(ts)$ for all $t \in S$. A mean μ on S is called left (right) invariant if $\mu(f_s) = \mu(f)$ $(\mu(f^s) = \mu(f))$ for all $f \in m(S)$ and $s \in S$. An invariant mean is a left and right invariant mean. A semigroup which has a left(right) invariant mean is called left(right) amenable. A semigroup which has an invariant mean is called amenable.

The following results are consequence of [9]

LEMMA 2.1. *Let E be a reflexive Banach space and let S be a set. Suppose that $\{x_s : s \in S\}$ is a bounded subset of element of E. Then, for a mean on S, we can obtain an* *element* x_0 *in* E *such that*

$$\mu_s(x_s,\ x^*)=(x_0,\ x^*)$$

for all $x_*\in E_*$.

LEMMA 2.2. *Let* E *be a real reflexive Banach space, let* S *be a right amenable semigroup and let* μ *be a right invariant mean on* S. *Suppose that* $\{x_t : t\in S\}$ *is a bounded subset of elements of* E. *Then, the mean point* $x_0\in E$ *of* x_t *concerning* μ *is contained in* $\bigcap_{s\in S}\overline{co}\{x_{ts} : t\in S\}$.

Recall that the norm of E is said to be Gateaux differentiable (and E is said to be smooth) if $\lim_{t\to\infty}(|x+ty|-|x|)/t$ exists for each x and y in $U=\{z\in E : |z|=1\}$. It is said to be uniformly Gâteaux differentiable if for each y in U, this limit is approached uniformly as x varies over U. The norm is said to be Fréchet differentiable if for each x in U this limit is attained uniformly for y in U.

3. Asymptotic behavior

We now study the mean points of $J_t x/t$ and $S(t)x/t$ concerning an invariant mean on $(0,\infty)$. Let E be a Banach space and let $A\subset E\times E$ be an accretive operator that satisfies the range condition. Then we know that for each $x\in cl(D(A))$, $|A_t x|$ is monotone nonincreasing in t and further by[6]

$$\lim_{t\to\infty}|A_t x|=\lim_{n\to\infty}|J_t x/t|=d(0,R(A)),$$

Where $d(0,R(A))=\inf\{|y| : y\in R(A)\}$

We shall need the following two known lemmas (cf. [2] and [8]).

LEMMA 3.1. E^* has a Fréchet differentiable norm if and only if E is reflexive and strictly convex, and has the following property: if the weak $\lim_{n\to\infty} x_n=x$ and $|x_n|\to|x|$, then

$\{x_n\}$ converges strongly to x.

LEMMA 3.2. *E^* has a Fréchet differentiable norm if and only if for any convex set $K\subset E$, every sequence $\{x_n\}$ in K such that $|x_n|$ tends to $d(0,K)$ converges.*

THEOREM 3.3. *Let E be a Banach space, $A\subset E\times E$ an accretive operator that satisfies the range condition, A_t the Yosida approximation of A, x a point in $cl(D(A))$, and v_t^{**} the natural image of $A_t x$ in E^{**}. If $d=(0,R(A))$, then $d=d(0,\overline{co}\{v_t^{**}\})$ for every $x\in cl(D(A))$ and there exists an element x^{**} with $|x^{**}|=d$ such that $x^{**}\in\overline{co}\{v_t^{**}\}$ for every $x\in cl(D(A))$*

PROOF. Let $x\in cl(D(A))$. Then, since $|A_t x|$ is monotone nonincreasingin t and $|A_t y|\leqq|Ay|$ for all $y\in D(A)$ and $t>0$, we have that $\{A_t x\}$ is bounded. Since v_t^{**} is the natural image of $A_t x$, by Lemma 2.1 and 2.2 there exists $x_0^{**}\in\overline{co}\{v_t^{**}\}$ such that $\mu_t(v_t^{**}, x^*)=(x_0^{**},x^*)$ for every $x^*\in E^*$, where μ is an invariant mean on $(0,\infty)$. For $j_0\in J(x_0^{**})$, where J is the duality mapping of E. We have

$$|x_0^{**}|^2=(x_0^{**}, j_0)=\mu_t(v_t^{**}, j_0)\leqq\mu_t|v_t^{**}||j_0|$$
$$=d|j_0|=d|x_0^{**}|.$$

Hence, $|x_0^{**}|\leqq d$.

on the other hand, we know that $(v_s^{**}, j_t)\geqq|v_t^{**}|$ for all $j_t\in J(v_t^{**})$ and t, $s\in(0,\infty)$ with $t>s>0$ (see[7]). Let $s\in S$ and let a subnet $\{j_{t_\alpha}\}$ of $\{j_t\}$ converges to $j\in E^{**}$. Then we obtain

$$(v_t^{**},j)\geqq d^2 \text{ for every } s\in S. \tag{3.1}$$

Hence we have $(x_0^{**},j)\geqq d^2$. Since $|j|\leqq\underline{\lim}_{\alpha}|j_{t_\alpha}|=\lim_{t\to\infty}|v_t^{**}|=d$, we have $d^2\geqq|x_0^{**}|\,|j|\geqq(x_0^{**}, j)\geqq d^2$ and hence $|x_0^{**}|=|j|=d$. From (3.1), we also obtain $(z^{**}, j)\geqq d^2$ for every $z^{**}\in co\{v_t^{**}\}$ and hence

$$|z^{**}|d=|z^{**}||j|\geqq(z^{**},\ j)\geqq d^2.$$

So, we have $|z^{**}|\geqq d$ and hence $d=d(0,\ \overline{co}\{v_t^{**}\})$.

Let $y\in cl(D(A))$ and y^{**} be a mean point of w_t^{**} concerning μ, where w_t^{**} natural mapping of A_ty. Then for $j\in J(x^{**}-y^{**})$, we have

$$\begin{aligned}|x^{**}-y^{**}|^2=(x^{**}-y^{**},\ j)&\\ &=\mu_t(v_t^{**}-w_t^{**},\ j)\\ &\leqq\mu_t|A_tx-A_ty|\ |j|\\ &\leqq\mu_t(\frac{2}{t}|x-y|)\ |j|\\ &=0\end{aligned}$$

and hence $x^{**}=y^{**}$. This observation the following corollary.

COROLLARY 3.4. *Let E be a Banach space, $A\subset E\times E$ an accretive operator that satisfies the range condition. J_t its resolvent, x a point in $cl(D(A))$ and u_t^{**} the natural image of J_tx in E^{**}. If E^* is smooth, and $d=d(0,R(A)$ Then th weak-star $\lim_{t\to\infty} u_t^{**}$ exists and is independent of $x\in cl(D(A))$.*

PROOF. Since E^* is smooth, hence J^* is single valued. So that $u_t^{**}=J^*(z(x)$ is single.

THEOREM 3.5. *Let E be a Banach space, $A\subset E\times E$ an accretive operator that satisfies the range condition, J_t the resolvent of A, and $d=d(0,R(A))$*

(a) *If E is reflexive and strictly convex, then the weak $\lim_{t\to\infty} J_tx/t$ exists for each x in $cl(D(A))$ (and its norm equals d).*

(b) *If E^* is Frechet differentiable norm, then the strong $\lim_{t\to\infty} J_tx/t$ exists.*

PROOF. Part (a)follows from Lemma 3.1.

Part (b) follows from Lemma 3.2 and part (a) because

$\lim\limits_{t\to\infty}|J_t x|/t=d$.

THEOREM 3.6. *Let E be a Banach space, $A\subset E\times E$ an accretive operator that satisfies the range condition, S the semigroup generated by $-A$, x a point in $cl(D(A))$ and w_t^{**} the natural image of $(x-S(t)x)/t$ in E^{**}. If $d=d(0,R(A))$, then $d=d(0, \overline{co}\{w_t^{**}\})$ to every $x\in cl(D(A))$ and there exists an element x^{**} with $|x^{**}|=d$ such that $x^{**}\in\overline{co}\{w_t^{**}\})$ for every $x\in cl(D(A))$.*

PROOF. Let $x\in cl(D(A))$. Then since $\lim\limits_{t\to\infty}|(x-S(t)x)/t|=d$ by [7] and further $\lim\limits_{t\to 0+}|(y-S(t)y)/t|\leqq||Ay||$ for all $y\in D(A)$, we have that $\{(x-S(t)y)/t\}$ is bounded. Also, by Lemma 2.1 and Lemma 2.2, there exists $x_0^{**}\in\overline{co}\{w_t^{**}\}$ such that $\mu_t(w_t^{**}, x^*)=(x_0^{**}, x^*)$ for every $x^*\in E^*$, where μ is an invariant mean on $(0,\infty)$. For $j_0\in J(x_0^{**})$, where J is the duality mapping of E, we have

$$\begin{aligned}|x_0^{**}|^2&=(x_0^{**}, j_0)=\mu_t(w_t^{**}, j_0)\leqq\mu_t|w_t^{**}||j_0|\\&=d|j_0|=d|x_0^{**}|\end{aligned}$$

Hence, $|x_0^{**}|\leqq d$. On the other hand, we know from [7] that for each $x\in cl(D(A))$, there is a functional $j\in E^*$ such that $(w_t^{**}, j)\geqq d^2$. So we have $(x_0^{**}, j)\geqq d^2$ and hence $|x_0^{**}|\geqq d$. Therefore $|x_0^{**}|=d$. Since $(w_t^{**},j)\geqq d^2$ for every $t\in S$, we also have $|z^{**}|\geqq d$ for every $z^{**}\in \overline{co}\{w_t^{**}\}$. Then we obtain $d=d(0, \overline{co}\{w_t^{**}\})$. Since $S(t)$ is nonexpansive, we have $x_0^{**}\in\overline{co}\{w_t^{**}\}$ for every $x\in cl(D(A))$ as in proof of Theorem 3.3. Thus we obtain the following Corollary.

COROLLARY 3.7. *Let E be a Banach space, $A\subset E\times E$ an accretive operator that satisfies the range condition, S the semigroup generated by $-A$, x a point in $cl(D(A))$ and w_t^{**} the natural image of $S(t)x/t$ in E^{**}. If E^* is* *smooth, and* $d=d(0, R(A))$. *Then the weak-star* $\lim_{t\to\infty} w_t^{**}$ *exists and is indenpendent of* $x\in cl(D(A))$.

THEOREM 3.8. *Let* E *be a Banach space,* $A\subset E\times E$ *an accretive operator that satisfies the range condition,* S *the semigroup generated by* $-A$, *and* $d=d(0, R(A))$.

(a) *If* E *is reflexive and strictly convex, then the weak* $\lim_{t\to\infty} S(t)x/t$ *exists for each* $x\in cl(D(A))$ (*and its norm equals* d)

(b) *If* E^* *is Fréchet differentiable norm, then the strong* $\lim_{t\to\infty} S(t)x/t$ *exists.*

PROOF. Part (a) follows form Lemma 3.1.

Part (b) follows from Lemma 3.2 and Part(a) because $\lim_{t\to\infty} |S(t)x/t|=d$.

THEOREM 3.9. *Let* C *be a closed subset of a Banach space* E *and* $T: C\to C$ *a nonexpansive mapping. Assume that* $A=I-T$ *satisfies the range condition* x *belong to* C, *and* u_n^{**} *the natural image of* $(x-T^n x)/n$ *in* E^{**}. *If* $d=d(0, R((A))$, *then* $d=d(0, \overline{co}\{u_n^{**}\})$ *to every* $x\in C$ *and there exists an element* x_0^{**} *with* $|x_0^{**}|=d$ *such that* $x_0^{**}\in \overline{co}\{u_t^{**}\}$ *for every* $x\in C$.

PROOF. Let $x\in C$. Then by [6], we know $\lim_{n\to\infty} |(x-T^n x)/n|=d$. So for a mean μ, there exists $x_0^{**}\in\overline{co}\{u_n^{**}\}$ such that $\mu_t(u_n^{**}, x^*)=(x_0^{**}, x^*)$ for every $x^*\in E^*$. For this point x_0^{**}, we have $|x_0^{**}|\leqq d$. Further from [7] we know that for each $x\in C$ there is a functional $j\in E^*$ with $|j|=d$ such that $(u_n^{**}, j)\geqq d$ for all $n\geqq 1$. So we have $(x_0^{**}, j)\geqq d^2$ and hence $|x_0^{**}|\geqq d$. Therefore $|x_0^{**}|=d$. Since $(u_n^{**}, j)\geqq d^2$ for every $n\geqq 1$, we also prove $|z^{**}|\geqq d$ for every $z^{**}\in\overline{co}\{u_n^{**}\}$. Then we obtain $d=d(0, \overline{co}\{u_n^{**}\})$. Since T

is nonexpansive We have $x_0^{**}\in\overline{Co}\{u_n^{**}\})$ for every $x\in C$ as in the proof of Theorem 3.3.

COROLLARY 3.10. *Let C be a closed subset of a Banach space E and $T: C\to C$ a nonexpansive mapping. Assume that $I-T$ satisfies the range condition. Let x belong to C, and let u_n^{**} be the natural image of $T^n x/n$ in E^{**}. If E^* is smooth, then the weak-star* $\lim_{n\to\infty} u_n^{**}$ *exists.*

THEOREM 3.11. *Let C be a closed subset of a Banach space space E and $T: C\to C$ a nonexpansive mapping. Assume that $A=I-T$ satisfies the range condition and let $d=d(0, R(A))$.*

(a) *If E is reflexive and strictly convex, then the weak* $\lim_{n\to\infty} T^n x/n$ *exist for each x in C (and its norm equals d).*

(b) *If E^* has a Fréchet differentiable norm, then the strong* $\lim_{n\to\infty} T^n x/n$ *exists.*

PROOF. By Lemma 3.1 and 3.2.

Reference

[1] H. Brezis, "Operateurs Maximaux Monotones et Semigroups de Contractions dans les Espaces de Hilbert," North-Holland, Amsterdam, 1973.

[2] K. Fan and I. Glicksberg, Some geometric properties of the spheres in a normed linear space, Duke Math. J. 25 (1958), 553-568.

[3] E. Kohlberg and A. Neyman, Asymptotic behavior of nonexpansive mappings in uniformly convex Banach space, preprint.

[4] E. Kohlberg and A. Neyman, Asymptotic behavior of nonexpansive mappings in normed linear space, Israel J. Math. in press.

[5] A. Pazy, Asymptotic behavior of contraction in Hiblert space, Israel J. Math. 9 (1971), 235-240.

[6] S. Reich, On the asymptotic behavior of nonlinear semigroups and the range of accretive operators, J. Math. Anal. Appl., 79 (1981), 113-126.

[7] ———, On the asymptotic behavior of nonlinear semigroups and the range of accretive operators II, J. Math. Anal. Appl., 87 (1982) 134-146.

[8] ———, Asymptotic behavior of semigroups of nonlinear contractions in Banach space, J. Math. Anal. Appl. 53 (1976), 277-293.

[9] W. Takahashi, The asymptotic behavior semigroups and invariant means, to appear in J. Math. Anal. Appl.

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