THE JOURNAL OF THE KOREAN ASTRONOMICAL SOCIETY
19: 63~68, 1986

CAUSTIC AND IMAGE PROPERTIES OF GRAVITATIONALLY BENDING LIGHT RAYS

CHANG, KYONGAE
Department of Physics and Optical Engineering, Chongju University
(Received November 15, 1986)

ABSTRACT

In this paper we deal with the orientation and the deformation of the circular light bundle passing in a static bounded gravitational field. The properties of caustic of the gravitational lens are discussed.

I. INTRODUCTION

As being confirmed the double QSO 0957+561 A, B as an evidence of the result of gravitational lens action of a massive galaxy in a cluster of galaxies (Young et al. 1980, 1981), the attention to the gravitational lens effect has been reviewed To search for more observational evidences of the effect, many authors have paid their efforts to construct more reasonable theoretical model of the gravitational lens, for instance, different types of the universe as a whole (see, e.g. Petrosian and Salpeter 1968), spheroidal or ellipsoidal model of galaxy (Bourassa et al. 1973). Such attempts, in consequence, have led to complexities in the formulation of the gravitational lens equations governing the mapping from the light source to the observer, and vice versa.

The linear approximation of superposed gravitatonal field has been introduced to simplify such a complexity of the gravitational lens equations, which led to easily solve the two-body gravitational lens model (Chang and Refsdal 1984). Further, it has been applied to n-body gravitational lens problem (Kayser et al. 1986).

It is well known that the number of image of the light source and its flux could be changed by the gravitational lens effect. The possibility to detect observational evidences of the gravitational lens effect is directly related to the flux changes of lensed images. Various properties of images, especially the number of images, have been studied in detail by many authors, for example, multiple structure of images, fine sub-structure of images (Bourassa et al. 1975; Chang and Refsdal 1984). In previous works, however, the orientation of the images in the image plane has been less discussed, because of its unimportant role to intensification or to deamplification of the images.

We derive the geometrical properties of the lensed images in framework of geometrical optics. In Section II caustic will be discussed. We also correct the statement on the caustic made by Ohanian (1983), which would be one of the main purporses of this paper. Orientation and deformation of the images will be reviewed in Sections III and IV. In concluding we present the results in Section V.

II. CAUSTIC

The trajectory of any light ray emitted at the light source (x', y'), passing close to the gravitating body with an impact parameter $b(\xi, \eta)$, and arriving at the observer (x,y), is given by the so-called gravitatonal lens equation. We assume that the three planes, $(x'$-$y')$-, $(\xi$ - $\eta)$-, and $(x$-$y)$-plane, are laid plane parallel. The mapping between these three planes will then be governed by the derivatives of the gravitational lens equation. That is, for a fixed observer at (x,y),

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = J \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}, \tag{1a}$$

with

$$J = \left(\frac{\partial(x,y)}{\partial(\xi,\eta)} \right), \tag{1b}$$

and for a fixed point source at (x',y') (or a fixed infinitesimal source element, when the source is considered to be extended one), we have

$$\begin{pmatrix} dx' \\ dy' \end{pmatrix} = J' \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}, \tag{2a}$$

with

$$J' = \left(\frac{\partial(x',y')}{\partial(\xi,\eta)} \right), \tag{2b}$$

where J and J' are the transformation matrixes. Equations (1) and (2) are general, no matter what type of the gravitational lens model we are under consideration. The caustic in the observer- and the source-plane are determined by

$$|J| = 0, \tag{3}$$
$$|J'| = 0, \tag{4}$$

respectively, where symbol $|\ \ |$ denotes the determinant. We may have

$$|J| = |J'|, \tag{5}$$

only when we use the normalized transformaiton equation. In other words, the lengths in three planes (source, deflector, observer) for mathematical convenience can be normalized by different units for a given set of the transformation equations. From equation (5) we can easily see that the caustics in the observer-and the source-plane are exactly same in shape and size in the normalized coordinates. However, it is worthwhile to note that if we convert those normalized units into the true dimensions, we shall still have exactly the same shape of the caustic in both planes, but *not in size* even in a non-expanding universe. The sizes of the caustics in both planes shall differ by a factor of $\lambda/(L-\lambda)$, where L and λ are the affine distances between the source and the observer and between the deflector and the observer. That is, the caustic in the source plane is larger (or smaller) by a factor of $\lambda/(L-\lambda)$ as compared with the caustic in the (x-y) plane (compare with Ohaninan's comment on page 553 right after eq.(12), Ohanian 1983). It is clear then that higher intensification of images will be observed, when the observer crosses the caustic surface in his plane.

III. ORIENTATION

We assume that the intrinsic light beam is of a circular cross section. From the transformation equation (eqs. (1) – (2)) we obtain the information on the distortion of the circular beam as travelling through the deflector plane to the observer. Eigenvalues and eigen vectors of matrix J in equation (1b) represent the amount of distortion and the orientation of the deflected light beam. By solving the matrix

$$\begin{vmatrix} \frac{\partial x}{\partial \xi} - \delta & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} - \delta \end{vmatrix} \qquad (6)$$

we obtain the eigenvalues of δ_+ and δ_-, which represent the deformation size along the two dimensional rectangular axes centred at the centre of the cross section of the light beam. In typical non-transparent gravitational lens problem, it is usual to have the elongation of the circular beam due to the gravitational lens effect.

IV. DEFORMATION

We introduce a set of new coordinate systems centred at the origin of the old coordinates (ξ, η) and (x,y), denoting the new coordinates by asterisk; such as (ξ^*, η^*) and (x^*, y^*). We also make use of the rotation matrix A and the Dehnung's matrix B,

$$A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \qquad (7)$$

$$B = \begin{pmatrix} \delta_+ & 0 \\ 0 & \delta_- \end{pmatrix} \qquad (8)$$

Then, the following transformation is available between the two coordinate systems. That is,

$$\begin{pmatrix} d\xi^* \\ d\eta^* \end{pmatrix} = A \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} dx^* \\ dy^* \end{pmatrix} = A \begin{pmatrix} dx \\ dy \end{pmatrix}, \tag{10}$$

which yields

$$A \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dx^* \\ dy^* \end{pmatrix} = J^* \begin{pmatrix} d\xi^* \\ d\eta^* \end{pmatrix} = J^* A \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}, \tag{11}$$

where

$$J^* = \left(\frac{\partial (x^*, y^*)}{\partial (\xi^*, \eta^*)} \right) = \begin{pmatrix} \alpha^* & \beta^* \\ \delta^* & \gamma^* \end{pmatrix}. \tag{12}$$

We then have

$$\frac{\partial x}{\partial \xi} - \frac{\partial y}{\partial \eta} = (\alpha^* - \delta^*) \cos 2\phi + (\beta^* + \gamma^*) \sin 2\phi, \tag{13}$$

$$\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} = (\alpha^* - \delta^*) \sin 2\phi - (\beta^* + \gamma^*) \cos 2\phi, \tag{14}$$

where

$$\alpha^* - \delta^* = 2 \cos\phi, \tag{15}$$

$$\beta^* + \gamma^* = 2 \sin\phi, \tag{16}$$

From equations (7) – (16) we obtain

$$\tan \phi = \frac{\dfrac{\partial x}{\partial \eta} + \dfrac{\partial y}{\partial \xi}}{\dfrac{\partial x}{\partial \xi} - \dfrac{\partial y}{\partial \eta}} \tag{17}$$

The image would then be oriented with an angle $\phi^* = \phi/2$, which is the angle between the major axes laid on the centre of the cross section of the light bundle before and after undergoing the gravitational light deflection. In general the diagonal terms of the transformation matrix $\partial x/\partial \eta$ and $\partial y/\partial \xi$ are equal. We may write

$$\phi^* = \tan^{-1} \frac{\dfrac{\partial x}{\partial \eta}}{\dfrac{\partial x}{\partial \xi} - \dfrac{\partial y}{\partial \eta}} \tag{18}$$

or

$$\phi^* = \tan^{-1} \frac{\dfrac{\partial y}{\partial \xi}}{\dfrac{\partial x}{\partial \xi} - \dfrac{\partial y}{\partial \eta}}. \tag{19}$$

V. RESULTS

In Table 1 the deformation parameters δ_+ and δ_- (eigenvalues of the transformation matrix eq. (1b)) are summarized with respect to the different types of galaxies acting as a gravitational lens. The symbols appeared in Table 1 are of the following meanings:

$$D = \frac{4G}{c^2} \frac{(L-\lambda)\lambda}{L} (1 + z), \tag{20}$$

where G is the gravitational constant, c, light velocity, and z, the red shift of the galaxy acting as a gravitational lens.

$$\widetilde{M} = \int_0^b 2\pi\, \sigma(r)\, b\, db\ ;\ \overline{M} = \pi\, \sigma(r)\, b^2, \tag{21}$$

where $\sigma(r)$ is the surface density at radial distance r from the centre of the galaxy.

$$\varepsilon = \frac{<\varepsilon^*>}{<\sigma_T>}, \tag{22}$$

with $<\sigma_T>$ the total average surface mass density around the light rays; σ^* the average surface mass density in "stars" (perturbing masses) in the region around the light rays under consideration.

"Empty light cone" effect has been discussed in detail by Zel'dovich (1964), later by Refsdal(1970). Note that all mass distributions (also nonsymmetric ones) with an empty light cone can be reduced to the point mass case, i.e. $|\delta_+| = |\delta_-|$ by an appropriate coordinate transformation.

The discussions and the equations given above all are general. No matter what complications are employed to the transformation equations, the basic property of diagonal symmetric transformation matrix should not be affected.

Table 1. Deformation of Lensed Image

Type of the gravitational lens	Extension along ξ −axis: δ_+	Contraction along η −axis: δ_-
Point mass with a mass M	$M\,D$	$-M\,D$
Extended mass	$(\widetilde{M} - 2\,\overline{M})\,D$	$-\widetilde{M}\,D$
Axial symmetric mass	$(\widetilde{M} - (2 - \varepsilon)\overline{M})\,D$	$-(\widetilde{M} - \varepsilon\overline{M})\,D$
Axial symmetric extended mass with empty light cone	$(\widetilde{M} - \overline{M})\,D$	$-(\widetilde{M} - \overline{M})\,D$

REFERENCES

Bourassa, R. R., Kantowski, R. and Nortn, T. D. 1973, Ap. J., **185**, 747.
Bourassa, R. R., and Kantowski, R. 1975, Ap. J., **195**, 13.
Chang, K., and Refsdal, S. 1984, Astr. Ap., **132**, 168.
Kayser, R., Refsdal, S., and Stabel, R. 1986, Astr. Ap., **166**, 36.
Ohanian, H. C. 1983, Ap. J., **271**, 551.
Petrosian, V., and Salpeter, E. E. 1968, Ap. J., **151**, 411.
Refsdal, S. 1970, Ap. J., **159**, 357.
Young, P., Gunn, J. E., Kristian, J., Oke, J. B., and Westphal, J. A. 1980, Ap. J., **241**, 507.
Young, P., Gunn, J. E., 1981, Ap. J., **244**, 736.
Zel'dovich, Ya. B. 1964, Soviet Astronomy, **8**, No. 1, 13.