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A Study on the Bayes Estimator of $\theta = Pr(Y < X)$

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ABSTRACT

We study the Bayes estimation procedure of $\theta = P_r(Y < X)$ when the experiment is terminated before all of the items on the test have failed and the failed items are partially replaced. Comparisons with the M.L.E., M.V.U.E. and Bayes estimator are made through Monte Carlo simulation.

I. INTRODUCTION

The problems of estimating $\theta = Pr(Y<X)$ have been studied in censored cases (Type 1, Type 2). The practice of terminating a life test with only partial information available is called censoring. Censoring procedure was first introduced by Epstein and Sobel (1953). Mendenhall and Harder (1958) and Boardmann and Kendell (1970) also considered the M.L.E. of parameter. Riley (1962) considered the cases where (a) items are replaced on failure (b) items are not replaced. Recently, Yeum and Kim (1984) obtained an estimator of θ in the censored cases.

In this paper, the Bayesian estimation procedure of θ is considered in partial replacement case. The partial replacement procedure may be derived as n items are placed on life test and the first k that fail are replaced but subsequent failures are not replaced. The experiment is terminated when the rth item fails so that, for a partial replacement, $k+1 \leqq r \leqq n+k$. The life times of items are assumed to be exponentially and independently distributed.

II. ESTIMATION OF θ

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be independently and identically distributed as

$$f_1(x) = \alpha e^{-\alpha x},\ \alpha > 0,\ x > 0. \quad \cdots\cdots (2.1)$$

and

$$f_2(y) = \beta e^{-\beta y},\ \beta > 0,\ y > 0. \quad \cdots\cdots (2.2)$$

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respectively. It is easily shown that if X and Y are independently distributed with densities given by (2.1) and (2.2), respectively, then

$$\theta = Pr(Y<X) = \int_0^\infty \int_0^\infty I(Y<X) dF_2(y) dF_1(x) = \frac{\beta}{\alpha+\beta}$$

where

$$I(Y<X) = \begin{cases} 1, & \text{if } Y<X \\ 0, & \text{otherwise.} \end{cases}$$

We shall assume a quadratic loss function and shall imploy conjugate prior distributions for α and β

$$g_1(\alpha) \propto \alpha^{\gamma_1 - 1} e^{-\delta_1 \alpha}, \qquad \gamma_1,\ \delta_1 > 0. \quad \cdots\cdots\cdots (2.3)$$

and

$$g_2(\beta) \propto \beta^{\gamma_2 - 1} e^{-\delta_2 \beta}, \qquad \gamma_2,\ \delta_2 > 0. \quad \cdots\cdots\cdots (2.4)$$

On making the transformation,

$$u_i = \begin{cases} n(x_{i+1} - x_i), & 0 \leqq i \leqq k \\ (n-i+k)(x_{i+1} - x_i), & k+1 \leqq i \leqq r-1 \end{cases}$$

where $x_0 = 0$, it can be seen that the $u_i; i=0,\ldots,r-1$ are independently and identically distributed with common density function $\alpha\ e^{-\alpha u}$ Hence, the likelihood function for α is

$$L(\alpha) = \alpha^r \exp(-\alpha\ t_1)$$

where

$$t_1 = \sum_{i=0}^{r-1} u_i = \sum_{i=1}^{r-k_1-1} (x_{k_1+i}) + (n-r+k_1+1)x_r$$

Thus the posterior p.d.f. for α is given by

$$\Pi_1(\alpha) \propto \frac{\alpha^{r+\gamma_1-1}\ e^{-\alpha(t_1+\delta_1)}}{\Gamma(r+\gamma_1)\,(t_1+\delta_1)^{-(r+\sigma_1)}}$$

Similarly,

$$\Pi_2(\beta) \propto \frac{\beta^{s+\gamma_2-1}\ e^{-\beta(t_2+\delta_2)}}{\Gamma(s+\gamma_2)(t_2+\delta_2)^{-(s+\delta_2)}}$$

where

$$t_2 = \sum_{j=1}^{s-1} v_1 = \sum_{j=1}^{r-k_2-1} (y_{k_2+j}) + (m-s+k_2+1)y_s.$$

Forming the joint posterior p.d.f. of α and β and letting $\theta = \frac{\beta}{\alpha+\beta}$ and $w = \alpha+\beta$, then we have the joint posterior p.d.f. of θ and W as follows;

$$f(\theta, w) \propto w^{r+s+\gamma_1+\gamma_2-1}\ \theta^{s+\gamma_2-1}\ (1-\theta)^{r+\gamma_1-1} \qquad \exp\left[-w(t_1+\delta_1)(1-c\,\theta)\right]$$

where

$$c = 1 - \frac{t_2+\delta_2}{t_1+\delta_1}$$

Thus, the marginal posterior p.d.f. of θ is given by

$$f(\theta) \propto \theta^{s+\gamma_2-1}(1-\theta)^{r+\gamma_1-1}\,\Gamma(r+s+\gamma_1+\gamma_2)\,\left[(t_1+\delta_1)(1-c\,\theta)\right]^{-(r+s+\gamma_1+\gamma_2)} \quad \cdots\cdots\cdots (2.5)$$

Therefore, we have the Bayes estimator of θ as follows;

(1) if $|c| < 1$,

$$\theta^* = E(\theta|x, y) = \frac{s+\gamma_2}{r+s+\gamma_1+\gamma_2}\ (1-c)^{s+\gamma_2}\ {}_2F_1(r+s+\gamma_1+\gamma_2,\ s+\gamma_2+1 : r+s+\gamma_1+\gamma_2+1 : c)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots (2.6)$$

where

$${}_2F_1(a, b : c : x) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\ \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt$$

(2) if $c \leqq -1$

$$\theta^* = \frac{s+\gamma_2}{r+s+\gamma_1+\gamma_2}(1-c)^{-(r+\gamma_1)}\ {}_2F_1(r+s+\gamma_1+\gamma_2,\ r+\gamma_1+1 : r+s+\gamma_1+\gamma_2+1 : c)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots (2.7)$$

By the transformation $\rho = \dfrac{1-\theta}{1-c\theta}$ in (2.5), the p.d.f. of ρ is given by

$$f(\rho) \propto \rho^{r+\gamma_1-1}(1-\rho)^{s+\gamma_2-1},\ 0<\rho<1.$$

Thus, for $0<\rho_1<\rho_2<1$.

$$Pr(\rho_1<\rho<\rho_2) = I(\rho_2 : r+\gamma_1,\ s+\gamma_2) - I(\rho_1 : r+\gamma_1,\ s+\gamma_2)$$

where $I(\rho; s, r) = \int_0^\rho B(s, r)$ denotes the p.d.f. of a beta distribution with parameters s and r.
Hense, we obtain the confidence limit on θ

$$Pr\left(\frac{1-\rho_2}{1-c\rho_2} < \theta < \frac{1-\rho_1}{1-c\rho_1}\right) = I(\rho_2 : r+\gamma_1,\ s+\gamma_2) - I(\rho_1 : r+\gamma_1,\ s+\gamma_2) \quad \cdots\cdots\cdots (2.8)$$

In order to obtain the bound on the Bayes estimator, we note the following. If $z^* = \frac{\alpha}{\beta} > 0$, then

$$E(z^*) = \int_0^\infty \int_0^\infty z^* \pi\ (\alpha, \beta) d\alpha\, d\beta$$

$$= \frac{r+\gamma_1}{s+\gamma_2-1}\ (1-c)$$

Similarly, let $z = \frac{\alpha}{\beta} > 0$. Then $E(z) = \frac{r+\gamma_1-1}{s+\gamma_2}(1-c)$. If we define $\theta = h(z^*) = \frac{1}{1+z^*}$, the $h(z^*)$ is convex function of z^*. Also, we define $\theta = g(z) = (1+z^{-1})^{-1}$, then $g(z)$ is a concave function of z. By Jenson's inequality, we obtain following result;

$$h[E(z^*)] = [1 + \frac{x+\gamma_1}{s+\gamma_2-1}\ (1-c)]^{-1}$$

$$\leqq E(\theta) = \theta_1^*$$

$$\leqq [1 + \frac{r+\gamma_1-1}{s+\gamma_2}\ (1-c)]^{-1}$$

$$= g\,[E(z)]$$

As an alternative to the use of the conjugate priors of (2.3) and (2.4), we may use the vague priors as following:

$$g_1(\alpha) = \frac{1}{\alpha^a} \qquad \text{and} \qquad g_2(\beta) = \frac{1}{\beta^b}$$

Then it is easily seen that, for these priors, the results corresponding to those obtained using conjugate priors, may be obtained by putting $\delta_1 = \delta_2 = 0$, $\gamma_1 = 1-a$ and $\gamma_2 = 1-b$ in (2.3)-(2.9), and substituting

$$c^* = \frac{t_1 - t_2}{t_1}$$

for c in the previous expressions. In particular bounds on $E(\theta)$ for this case are

$$[1 + \frac{r-a+1}{s-b}(1+c^*)]^{-1} < E(\theta) < [1 + \frac{r-a}{s-b+1}(1-c^*)]^{-1} \quad \cdots\cdots\cdots\cdots\cdots (2.10)$$

In (2.10), we know that the Bayes estimator is approximated by M.L.E. of θ for sufficiently large value of r and s.(see Yeum and Kim (1984)).

III. EMPIRICAL COMPARISON FOR MODERATE SIZED SAMPLES

In previous chapter, we derived the Bayes estimator. Yeum and Kim (1984) obtained the M.L.E. and M.V.U.E. of θ in censored cases.

In this section, we investigate their relative performance for a moderate sized sample through Monte Carlo simulation.

For fixed $k_1=k_2=0$, $r=s=10$ and $n=m=20$, estimates of the mean square error (M.S.E.) and bias are obtained from 2,000 trials with $Z=\frac{\beta}{\alpha}=1,2,3,4$ and 5. In each trial, $(n+m)$ random numbers U_i from an uniform distribution on (0, 1) are generated and they are transformed in to $X_i=-\log U_i, i\leqq n$, and $Y_i=-\frac{1}{z}\log U_i$, $i=n+1, n+2, \ldots, n+m$. From these data, the value of the M.L.E., M.V.U.E. and Bayes estimate of θ for vague priors are obtained. The estimated M.S.E.'s and biases of M.L.E., M.V.U.E. and Bayes E. appear in Table. Although M.V.U.E. is unbiased, its estimated bias is recorded for a check on the compution.

From Table, we know the following facts;

(1) The estimated M.S.E.'s are nearly equal each other.

(2) In all cases included in the study, the magnitude of $(\text{bias})^2$ is relatively negligible to the M S.E..

(3) The estimated M.S.E.'s of Bayes estimator are smaller than the others.

z	θ	Bias			M.S.E.		
		M.L.E.	M.V.U.E.	Bayes E.	M.L.E.	M.V.U.E.	Bayes E.
1	0 500	0.003	0.003	0.003	0.011	0.012	0.010
2	0.666	0.009	0.002	0.006	0.009	0.010	0.008
3	0.750	0 011	0.002	0.009	0.007	0.007	0.007
4	0.800	0.011	0.002	0.010	0.005	0.005	0.005
5	0.833	0.011	0.002	0.010	0.004	0.004	0.004

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