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A CHARACTERIZATION OF GROUPS AND LEFT GROUPS

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A semigroup S is said to be a left group if, for every couple a, $b \in S$ there exists a unique element x of S such that $xa=b$. It is well known that a semigroup S is a left group if and only if S is left simple and right cancellative; or, equivalently, S is regular and the set $E(S)$ of idempotent elements of S is a left zero semigroup. The present author [2] has recently proved that a semigroup S is a left group if and only if $B(ab)=B(a)$ holds for every couple a, b in S, where $B(a)$ is the principal bi-ideal of S generated by the element a of S. Another characterization by the author reads as follows.

LEMMA 1. *A semigroup S is a left group if and only if the multiplicative semigroup $B(S)$ of all bi-ideals of S is a left zero band.*

We need also the following criterion of this author [3].

LEMMA 2. *A semigroup S is a semilattice of left groups if and only if $B(S)$ is a left regular band.*

First we shall prove the following result.

THEOREM 1. *A semigroup S is a left group if and only if the condition**

$$BAL=B \tag{1}$$

holds for every left ideal L and for every couple A, B in $B(S)$.

PROOF. The necessity follows at once from Lemma 1. Conversely, if S is a semigroup having property (1) for every left ideal L and for all $A, B \in B(S)$, then (1) implies $B=B^2S$ for every bi-ideal B of S, whence every bi-ideal B of S is a right ideal of S. Next we show that S is regular. (1) implies $R=R^2S$ for any right ideal R of S. Hence it follows that $R \subseteq R^2$, and thus $R=R^2$. Therefore the bi-ideal semigroup $B(S)$ is a band, and S is regular. But S is a left duo regular semigroup, which is a semilattice of left groups. By our Lemma 2, for any left ideal L of S. we have $SLS=SL=L$. Hence $L=S$, by

* The Hungarian word BAL means LEFT in English.

(1), and S is left simple. Therefore (1) implies $R_1R_2S=R_1R_2=R_1$ for every couple R_1, $R_2 \in B(S)$. Finally, Lemma 1 implies that S is a left group.

THEOREM 2. *A semigroup S is a group if and only if the equality*

$$B=ABL \tag{2}$$

holds for every left ideal L and for all A, $B \in B(S)$.

PROOF. Necessity is trivial because of a group has no proper bi-ideal. Sufficiency: (2) implies $B=SBS$ for any bi-ideal B of S. Hence every bi-ideal B is a two-sided ideal of S. Now (2) implies $I=I^3 \subseteq I^2$ for every two-sided ideal I of S. Hence it follows that every two-sided ideal of S is globally idempotent and thus S is a regular duo semigroup. This means (cf. [4]), that S is a semilattice of groups. Then, for any ideal I of S,

$$S=ISI=IS=SI=I.$$

Therefore S is the only bi-ideal of S, whence it follows that S is a group, indeed.

Finally we formulate the left-right dual of Theorem 1.

THEOREM 3. *A semigroup S is a right group if and only if the condition $B=RAB$ holds for all A, $B \in B(S)$ and for every right ideal R of S.*

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