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ON SEMI-WEAKLY CONTINUOUS MAPPINGS

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1. Introduction

In 1961, N. Levine [1] introduced the concept of weakly continous mappings. P.E. Long and D.A. Carnahan [3] studied several properties of almost continous mappings in the sense of Singal [6]. T. Noiri [4] pointed out that the word "almost continuous" can be replaced by "weakly continuous" in some theorems of [3]. The purpose of this note is to introduce a new class of mappings called semi-weakly continuous mappings and investigate some properties analogous to those given in [4] concerning weakly continuous mappings.

2. Preliminaries

Let X be a topological space and S be a subset of X. A subset S is said to be *semi-open* [2] if there exists an open set U such that $U \subset S \subset \mathrm{Cl}(U)$, where $\mathrm{Cl}(U)$ denotes the closure of U. The complemet of a semi-open set is called *semi-closed*. The union of all semi-open sets of X contained in S is called the *semi-interior* of S and denoted by $s\mathrm{Int}(S)$. The intersection of all semi-closed sets of X containing S is called the *semi-closure* of S and denoted by $s\mathrm{Cl}(S)$. A mapping $f: X \to Y$ is said to be *weakly continous* [1] (resp. *almost continuous* [6]) if for each $x \in X$ and each open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset \mathrm{Cl}(V)$ (resp. $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$), where $\mathrm{Int}(S)$ is the interior of S.

3. Semi-weakly continuous mappings

DEFINITION 1. A mapping $f: X \to Y$ is called *semi-weakly continuous* (briefly *s.w.c.*) if for each $x \in X$ and each open set V containing $f(x)$ there exsits a semi-open set U containing x such that $f(U) \subset s\mathrm{Cl}(V)$.

A mapping $f: X \to Y$ is said to be *semi-continuous* [2] if for each open set V of Y, $f^{-1}(V)$ is semi-open in X. In [2, Theorem 12], it is known that a mapping $f: X \to Y$ is semi-continous if and only if for each $x \in X$ and each open set V containing $f(x)$ there exists a semi-open set U containing x such that $f(U) \subset V$. Therefore, every semi-continuous mapping is s.w.c., but the converse is not true as the following example shows.

EXAMPLE 1. Let X and Y be both the set of real numbers. Let τ be the usual topology for X and σ the cocountable topology for Y. Then the identity mapping $f:(X, \tau)\to(Y, \sigma)$ is semi-weakly continuous and not semi-continuous.

THEOREM 1. *A mapping $f:X\to Y$ is s.w.c. if and only if for every open set V in Y $f^{-1}(V)\subset s\mathrm{Int}(f^{-1}(s\mathrm{Cl}(V)))$.*

PROOF. Let $x\in X$ and V an open set containing $f(x)$. Then $x\in f^{-1}(V)\subset s\mathrm{Int}(f^{-1}(s\mathrm{Cl}(V)))$. Put $U=s\mathrm{Int}(f^{-1}(s\mathrm{Cl}(V)))$. Then U is semi-open and $f(U)\subset s\mathrm{Cl}(V)$. Conversely, let V be any open set of Y and $x\in f^{-1}(V)$. Then there exists a semi-open set U in X such that $x\in U$ and $f(U)\subset s\mathrm{Cl}(V)$. Therefore, we have $x\in U\subset f^{-1}(s\mathrm{Cl}(V))$ and hence $x\in s\mathrm{Int}(f^{-1}(s\mathrm{Cl}(V)))$. This proves that $f^{-1}(V)\subset s\mathrm{Int}(f^{-1}(s\mathrm{Cl}(V)))$.

THEOREM 2. *Let $f:X\to Y$ be a mapping and $g:X\to X\times Y$ be the graph mapping of f, given by $g(x)=(x, f(x))$ for every point $x\in X$. If g is s.w.c., then f is s.w.c.*

PROOF. Let $x\in X$ and V be any open set containing $f(x)$. Then $X\times V$ is an open set in $X\times Y$ containing $g(x)$. Since g is s.w.c., there exists a semi-open set U containing x such that $g(U)\subset s\mathrm{Cl}(X\times V)$. It follows from Lemma 4 of [5] that $s\mathrm{Cl}(X\times V)\subset X\times s\mathrm{Cl}(V)$. Sicne g is the graph mapping of f, we have $f(U)\subset s\mathrm{Cl}(V)$. This shows that f is s.w.c.

THEOREM 3. *If $f:X\to Y$ is a s.w.c. mapping and Y is Hausdorff, then the graph $G(f)$ is a semi-closed set of $X\times Y$.*

PROOF. Let $(x, y)\notin G(f)$. Then, we have $y\neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets W and V such that $f(x)\in W$ and $y\in V$. Since f is s.w.c., there exists a semi-open set U containing x such that $f(U)\subset s\mathrm{Cl}(W)$. Since W and V are disjoint, we have $V\cap s\mathrm{Cl}(W)=\phi$ and hence $V\cap f(U)=\phi$. This shows that $(U\times V)\cap G(f)=\phi$. It follows from Theorems 2 and 11 in [2] that $G(f)$ is semi-closed.

DEFINITION 2. By a *s.w.c. retraction*, we mean a s.w.c. mapping $f:X\to A$, where $A\subset X$ and $f|A$ is the identity mapping on A.

THEOREM 4. *Let $A\subset X$ and $f:X\to Y$ be a s.w.c. retraction of X onto A. If X is a Hausdorff space, then A is a semi-closed set in X.*

PROOF. Suppose that A is not semi-closed. Then there exists a point $x \in s\mathrm{Cl}(A)-A$. Since f is s.w.c. retraction, we have $f(x) \neq x$. Since X is Hausdorff, there exist disjoint open sets U and V such that $x \in U$ and $f(x) \in V$. Thus we get $U \cap s\mathrm{Cl}(V)=\phi$. Now, let W be any semi-open set in X containing x. Then $U \cap W$ is a semi-open set containing x and hence $(U \cap W) \cap A \neq \phi$ because $x \in s\mathrm{Cl}(A)$. Let $y \in (U \cap W) \cap A$. Since $y \in A$, $f(y)=y \in U$ and hence $f(y) \notin s\mathrm{Cl}(V)$. This gives that $f(W) \not\subset s\mathrm{Cl}(V)$. This contradicts that f is s.w.c. Hence A is semi-closed in X.

4. S-connected space

DEFINITION 3. A space X is said to be *S-connected* [7] if X can not be written as the disjoint union of two non-empty semi-open sets.

Every S-connected space is connected but the converse is not true as the following example shows.

EXAMPLE 2. Let $X=\{a, b, c\}$ and $\tau=\{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a connected space. However, it is not S-connected.

It is shown in Theorem 4 of [3] (resp. Theorem 3 of [4]) that connectedness is invariant under almost continuous (resp. weakly continuous) surjections. It is also known that S-connectedness is invariant under semi-continuous surjections. However, we have the following.

THEOREM 5. *If X is an S-connected space and $f: X \to Y$ is a s.w.c. surjection, then Y is connected.*

PROOF. Suppose that Y is not connected. Then there exist non-empty open sets V_1 and V_2 of Y such that $V_1 \cap V_2=\phi$ and $V_1 \cup V_2=Y$. Hence, we have $f^{-1}(V_1) \cap f^{-1}(V_2)=\phi$, $f^{-1}(V_1) \cup f^{-1}(V_2)=X$ and $f^{-1}(V_i) \neq \phi$ because f is surjective. By Theorem 1, we have

$$f^{-1}(V_i) \subset s\mathrm{Int}(f^{-1}(s\mathrm{Cl}(V_i))),\ i=1,\ 2.$$

Since V_i is open and closed, we obtain $f^{-1}(V_i) \subset s\mathrm{Int}(f^{-1}(V_i))$ and hence $f^{-1}(V_i)$ is semi-open for $i=1, 2$. This implies that X is not S-connected. Therefore Y is connected.

THEOREM 6. *If X is an S-connected space and $f: X \to Y$ is a semi-continuous mapping with the closed graph, then f is constant.*

PROOF. Suppose that f is not constant. There exist distinct points x_1, x_2 in X such that $f(x_1) \neq f(x_2)$. Since the graph $G(f)$ is closed and $(x_1, f(x_2)) \notin G(f)$, there exist open sets U and V containing x_1 and $f(x_2)$, respectively, such that $f(U) \cap V = \phi$. Since f is semi-continous, U and $f^{-1}(V)$ are disjoint non-empty semi-open sets. It follows from Theorem 17 of [7] that X is not S-connected. Therefore, f is constant.

COROLLARY (Thompson [7]). *Let X be irreducible. If $f: X \to Y$ is a continuous mapping with the closed graph, then f is constant.*

PROOF. Since every continuous mapping is semi-continuous, this is an immediate consequence of Theorem 17 in [7] and Theorem 6.

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REFERENCES

[1] N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, 68(1961), 44—46.

[2] ______, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 70(1963), 36—41.

[3] P.E. Long and D.A. Carnahan, *Comparing almost continuous functions*, Proc. Amer. Math. Soc., 38(1973), 413—418.

[4] T. Noiri, *On weakly continuous mappings*, Proc. Amer. Math. Soc., 46(1974), 120—124.

[5] ______, *A note on s-regular spaces*, Glasnik Mat., 13(33) (1978), 107—110.

[6] M.K. Singal and A.R. Singal, *Almost-continuous mappings*, Yokohama Math. J., 16 (1968), 63—73.

[7] T. Thompson, *Characterizations of irreducible spaces*, Kyungpook Math. J., 21(1981), 191—194.