Kyungpook Math. J.
Volume 25, Number 2
December, 1985

EXTENSIONS OF ANTI-DERIVATIONS TO MODULES OF QUOTIENTS

(Dedicate to professor Jae Kyu Lim on his sixtieth birthday)

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1. Introduction

Throughout the following, R will denote an associative ring with unit element 1 and R-Mod will denote the category of all unitary left R-modules and let $w: R \longrightarrow R$ be an involution (i.e. w is an endomorphism of R whose square is identity map). Then anti-derivation with respect to w of R is a mapping $d: R \longrightarrow R$ such that $d(a+b)=d(a)+d(b)$ and $d(ab)=d(a)b+w(a)d(b)$ for all elements $a, b \in R$ ([3]). If w is an identify map, then d is called an (ordinary) derivation.

If M is a unitary left R-module and if d is a fixed anti-derivation (with respect to w) on R then anti-derivation on M is a mapping $\bar{d}: M \rightarrow M$ satisfying the condition that $\bar{d}(m+n)=\bar{d}(m)+\bar{d}(n)$ and $\bar{d}(rm)=d(a)m+w(a)\bar{d}(m)$ for all element $m, n \in M$ and $r \in R$.

The purpose of the present paper is to show that any anti-derivation w.r. to w on a left R-module M can be extended to an anti-derivation w.r. to w on the module of quotients of M with respect any torsion theory on R-Mod relative to which M is torsionfree, using the method of J. Golan's method. In particular any anti-derivation w.r. to w on the ring R can be extended to an anti-derivation w.r. to w on the ring of quotients of R, uniquely.

2. Some preliminaries

Notation and terminanology concerning (hereditary) torsion theories on R-Mod will follow [1]. In particular, if τ is a torsion theory on R-Mod then a left ideal H of R is said to be τ-dense ideal in R if and only if the cyclic left R-module R/H is τ-torsion. If M is a left R-module then we denote $T_\tau(M)$ the unique largest submodule of M which is τ-torsion. If $E(M)$ is the injective hull of a left R-module M then we define the submodule $E_\tau(M)$ of $E(M)$ by $E_\tau(M)/M=T_\tau((E)/M)$. The module of quotients of M with respect to τ, denoted by $Q_\tau(M)$, is then defined to be $E_\tau(M/T_\tau(M))$. Note that, in particular, if M is τ-torsionfree then $Q_\tau(M)=E_\tau(M)$, and this is a left R-module containing M as a large submodule. In general, we have a canonical R-homomorphism from

M to $Q_\tau(M)$ obtained by composing the canonical surjection from M to $M/T_\tau(M)$ with the inclusion map into $Q_\tau(M)$.

If R_τ is the endomorphism ring of the left R-module $Q_\tau({}_RR)$ then $Q_\tau(M)$ is canonically a left R-module for every left R-module M and the canonical map $R\to R_\tau$ is a ring homomorphism The ring R_τ is called the ring of quotients or localization of R at τ. A torsion theory τ on R-Mod is said to be faithful if and only if R, considered as a left module over itself, is τ-torsionfree. In this case, R is canonically subring of R_τ.

Before enterning our discussion, we assume that any anti-derivations are related with a fixed involution w.

LEMMA 1. *For each q in $Q_\tau(M)$, the map $\alpha_{H,q}: H\to Q_\tau(M)$ defined by $h\to \bar{d}(w(h)q)-d(w(h))q$ is an R-module homomorphism, for every h in H.*

PROOF. Trivial.

The following lemmas can be found in [1].

LEMMA 2. *Let H be a τ-dense ideal in R, and let $\alpha_{H,q}$ be R-module homomorphism defined on H into $Q_\tau(M)$, then R/H is τ-torsion and there exist unique R-module homomorphism $\beta_{R,q}: R\to Q_\tau(M)$ which makes the diagram commutes.*

i.e.

$$\begin{array}{ccccc} o & \longrightarrow & H & \longrightarrow & R \\ & & {\scriptstyle \alpha_{H,q}}\downarrow & \swarrow {\scriptstyle \beta_{R,q}} & \\ & & Q_\tau(M) & & \end{array}$$

LEMMA 3. *Let H and K be τ-dense ideals of R then we have the following results.*

(1) *$H\cap K$ is τ-dense ideal.*
(2) *$(H:r)=\{a\in R \mid ar\in H\}$ is τ-dense ideal.*
(3) *Homomorphic image of H is τ-dense ideal.*

LEMMA 4. *Let H and K be τ-dense ideals of R and let $\alpha_{H,q}: H\to Q_\tau(M)$, and $\alpha_{K,q}: K\to Q_\tau(M)$ be defined as in the lemma 2. Then $\alpha_{H,q}$ and $\alpha_{K,q}$ define the same elements in $Q_\tau(M)$.*

3. Main theorems

THEOREM 5. *Let d be an anti-derivation on a ring R. Let τ be a torsion theory on R-Mod and let M be a τ-torsionfree left R-module on which we have defined* *an anti-derivation $\bar{d}$. Then there exists an anti-derivation $\bar{\bar{d}}$ defined on $Q_\tau(M)$ the restriction of which to M is $\bar{d}$.*

PROOF. If q is an element of $Q_\tau(M)$ then there exists a τ-dense left ideal H of R satisfying $Hq\leq M$. Define a function $\alpha_{H,q}: H\to Q_\tau(M)$ by setting $h\to \bar{d}(w(h))q)-d(w(h))q$. By the lemma 1, $\alpha_{H,q}$ is an R-homomorphism of left R-modules. Therefore by the lemma 2, we see that $\alpha_{H,q}$ extends uniquely to R-homomorphism from ${}_RR$ to $Q_\tau(M)$ and so there exists unique element $\bar{q}$ of $Q_\tau(M)$ satisfying the condition that $\alpha_{H,q}=h\bar{q}$ for all h in H. We now define a function $\bar{\bar{d}}: Q_\tau(M)\to Q_\tau(M)$ by setting $\bar{\bar{d}}(q)=\bar{q}$. This function is welldefined. Indeed, suppose that q is an element of $Q_\tau(M)$ and let H and K be τ-dense left ideals of R satisfying $Hq\leq M$ and $Kq\leq M$. Then $(H\cap K)q\leq M$ and $H\cap K$ is also τ-dense ideal in R. By the lemma 4, $\alpha_{H,q}$ and $\alpha_{K,q}$ define the same element $\bar{q}$.

Now we claim such $\bar{\bar{d}}$ is an anti-derivation on $Q_\tau(M)$ Indeed, let p and q be elements of $Q_\tau(M)$ and let r be an element of R. If H and J are τ-dense left ideals of R satisfying $Hp\leq M$ and $Jq\leq M$, then $K=H\cap J$ is τ-dense ideal of R such that $Kp\leq M$ and $Kq\leq M$. Moreover, for every element k of K we have

$$\begin{aligned}(k)\alpha_{K,\,p+q}&=\bar{d}(w(k)(p+q))-d(w(k))(p+q)\\&=\bar{d}(w(k)p)+\bar{d}(w(k)q)-d(w(k))p-d(w(k))q\\&=(k)(\alpha_{K,p}+\alpha_{K,q})\end{aligned}$$

and by the lemma 2, the uniqueness of extension, this implies that

$$\bar{\bar{d}}(p+q)=\bar{\bar{d}}(p)+\bar{\bar{d}}(q).$$

Similarly there exists a τ-dense left ideal H of R satisfying the condition that $Hrq\leq M$ and $Hq\leq M$. Then by the lemma 3, $(H:r)$ and $K=H\cap(H:r)$ are τ-dense left ideals of R, we therefore have a R-homomorphism from ${}_RK$ to ${}_RR$ given by $k\longrightarrow(k)\alpha_{K,rq}-(kw(r))\alpha_{K,q}$. For every element k of K, we see that

$$\begin{aligned}(k)\alpha_{K,rq}-(kw(r))\alpha_{K,q}&=\bar{d}(w(kr)q)-d(w(k))rq-\bar{d}(w(k)r)q+d(w(k)r)q\\&=-d(w(k))rq+d(w(k))rq+w^2(k)d(r)q=kd(r)q\end{aligned}$$

And by the uniqueness of the extension, this equation implies that $\bar{\bar{d}}(rq)-w(r)\bar{\bar{d}}(q)=d(r)q$, i.e. $\bar{\bar{d}}(rq)=d(r)q+w(r)\bar{\bar{d}}(q)$. Thus $\bar{\bar{d}}$ is an anti-derivation on $Q_\tau(M)$.

Note that $\bar{\bar{d}}$ restricts to $\bar{d}$ on $Q_\tau(M)$. Indeed m is an element of M then we can take R as a τ-dense ideal in R such that $Rm\leq M$ and for any element r of R we have $\bar{d}(rm)-d(r)m=d(r)m+w(r)\bar{d}(m)=w(r)\bar{d}(m)$. By the definition of $\bar{\bar{d}}$, $\bar{\bar{d}}(rm)-d(r)m=w(r)\bar{\bar{d}}(m)$ i.e. $\bar{\bar{d}}(m)=\bar{d}(m)$.

COROLLARY 6. *Let d be an anti-derivation on a ring R and let $\bar{d}$ be an anti-derivation defined on a left R-module M. Suppose that τ is a torsion theory on R-Mod satisfying the condition that $\bar{d}(T_\tau(M))\leq T_\tau(M)$. Then there exists an anti-derivation $\bar{\bar{d}}$ on $Q_\tau(M)$ in such a manner that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\ \ -\ \ } & Q_\tau(M) \\ \bar{d}\downarrow & & \downarrow \bar{\bar{d}} \\ M & \longrightarrow & Q_\tau(M) \end{array}$$

commutes.

PROOF. Define d^* on $M/T_\tau(M)$ by denotting, $d^*: m+T_\tau(M)\mapsto \bar{d}(m)+T_\tau(M)$, by the condition $\bar{d}(T_\tau(M))\leq T_\tau(M)$, such a map is welldefined. And $M/T_\tau(M)$ is τ-torsionfree left R-module by the theorem 5, this anti-derivation d^* can be extended to an anti-derivation $\bar{\bar{d}}$ on $Q_\tau(M)$ making the diagram commutes.

CORROLLARY 7. *Let τ be a faithful torsion theory on R-Mod and d be an anti-derivation on a ring R. Then there exists unique anti-derivation $\bar{\bar{d}}$ defined on R_τ the restriction of which to R is d.*

PROOF. The existension of anti-derivation $\bar{\bar{d}}$ follows from the theorem 5 and the fact that $Q_\tau(R)$ and R_τ are isomorphic, as left R-modules. To show the uniqueness, assume that d^* and f^* be anti-derivations defined on R_τ, and $d^*=f^*$ on R i.e. $(d^*-f^*)(h)=0$ for all $h\in R$ For all non-zero element q of R_τ there exists a τ-dense left ideal H of R satisfying the condition that $Hq\leq R$ and so for any element h in H we have $0=(d^*-f^*)(hq)=w(h)(d^*-f^*)(q)$, thus we have $w(H)(d^*-f^*)(q)=0$. Since $w(H)$ is τ-dense left ideals of R, this implies that $d^*(q)=f^*(q)$ for all $q\in R$.

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