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A SIMPLE PROOF OF KNOPP AND SINKHORN'S THEOREM

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Let J_n denote the n-square matrix all of whose entries are $\frac{1}{n}$ and

$$T_n=\left(\begin{array}{c|c} 0 & \frac{1}{n-1}\cdots\cdots\frac{1}{n-1} \\ \hline \frac{1}{n-1} & \\ \vdots & \frac{n-2}{n-1}J_{n-1} \\ \frac{1}{n-1} & \end{array}\right)$$

Let Ω_n denote the set of all n-square doubly stochastic matrices and

$$\partial\Omega_n=\{X=[x_{ij}]\in\Omega_n \mid x_{11}=0\}.$$

A matrix $A\in\partial\Omega_n$ is called a minimizing matrix on $\partial\Omega_n$ if $\mathrm{per}A\leqq\mathrm{per}X$ for all $X\in\partial\Omega_n$.

P. Knopp and R. Sinkhorn have proved the following

THEOREM [3]. *If $n\geqq 4$, T_n is the unique minimizing matrix on $\partial\Omega_n$.*

By using the limit process with one of their old results in [4], namely,

LEMMA [4]. *If $n\geqq 4$,* per$X$ *has a local strict minimum at T_n on the boundary of Ω_n, and hence on $\partial\Omega_n$.*

In this note, we shall provide an elementary combinatorial proof of the uniqueness part of the theorem without using their lemma. An n-square matrix is called fully indecomposable if it does not contain an $s\times t$ zero submatrix with $s+t=n$. For an n-square (0, 1) -matrix D, define

$$\Omega(D)=\{X\in\Omega_n \mid X\leqq D\}.$$

For an n-square matrix A, $A(i, j)$ will stand for the $(n-1)$-square matrix obtained from A by striking out the row i and the column j.

LEMMA 1. [Foregger, 2]. *Let $D=[d_{ij}]$ be an n-square, fully indecomposable* (0, 1)*-matrix, and let $A=[a_{ij}]$ be a minimizing matrix on $\Omega(D)$. If, for some i, j, $d_{ij}=1$, then*

$$\operatorname{per}A(i,\ j)=\operatorname{per}A \text{ if } a_{ij}>0,$$
$$\operatorname{per}A(i,\ j)\geqq\operatorname{per}A \text{ if } a_{ij}=0.$$

LEMMA 2. [Minc, 5]. *Let $D=[d_1,\ldots,d_n]$ and A be the same as in Lemma 1. If, for some $k\leqq n$, $d_1=\cdots=d_k$, then, for any $p\leqq k$, $A(J_p\oplus I_{n-p})$ is a minimizing matrix on $\Omega(D)$. A similar statement holds for rows.*

We can easily prove the following lemma as a corollary of Lemma 2.

LEMMA 3. *Let $n\geqq 3$, and let A be a minimizing matrix on $\partial\Omega_n$. Then, for any k, $p\leqq n-1$,*

$$(I_1\oplus J_k\oplus I_{n-k-1})A(I_1\oplus J_p\oplus I_{n-p-1})$$

is a minimizing matrix on $\partial\Omega_n$.

PROOF OF THEOREM. It is clear, by Lemma 3, that T_n is minimizing matrix on $\partial\Omega_n$. We are to show that T_n is the unique minimizing matrix on $\partial\Omega_n$.

Let $A=[a_{ij}]$ be a minimizing matrix on $\partial\Omega_n$. We claim, first, that for all $i=2,\ldots,n$,

$$a_{i1}\leqq\frac{1}{n-1}.$$

For, suppose not. Then, without loss of generality, we may assume that

$$a_{n1}>\frac{1}{n-1}.$$

Let $B=(I_1\oplus J_{n-2}\oplus I_1)A(I_1\oplus J_{n-1})$. Then, B is a minimizing matrix on $\partial\Omega_n$ by Lemma 3, which has the form

$$B=\left(\begin{array}{c|ccc} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \hline c & a & \cdots & a \\ \vdots & \vdots & & \vdots \\ c & a & \cdots & a \\ \hline a_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right).$$

Since $a_{n1}>\frac{1}{n-1}$, we have $c<\frac{1}{n-1}$, and hence $a>\frac{n-2}{(n-1)^2}$.

Thus

$$\begin{aligned}\operatorname{per}B(n,\ 1)&=(n-2)!\ a^{n-2}\\ &>(n-2)!\left(\frac{n-2}{(n-1)^2}\right)^{n-2}=\operatorname{per}T_n,\end{aligned}$$

contradicting the minimality of per$B$ by Lemma 1. So, the claim is proved.

Thus we have

$$a_{21}=\cdots=a_{n1}=\frac{1}{n-1},$$

since $\sum_{i=2}^{n} a_{i1}=1$. Similary we can show that

$$a_{12}=\cdots=a_{1n}=\frac{1}{n-1}.$$

Now, let's show that $a_{ij}=\frac{n-2}{(n-1)^2}$ for all i, $j=2,\dots,n$. Suppose not. Then, we may assume, without loss of generality, that $a_{nn}>\frac{n-2}{(n-1)^2}$. Let

$$C=(I_1\oplus J_{n-2}\oplus I_1)A(I_1\oplus J_{n-2}\oplus I_1).$$

Then C is a minimizing matrix on $\partial\Omega_n$ which looks like

$$C=\left(\begin{array}{c|ccc|c} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \hline \frac{1}{n-1} & d & \cdots & d & b \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{n-1} & d & \cdots & d & b \\ \hline \frac{1}{n-1} & b & \cdots & b & a_{nn} \end{array}\right).$$

Since $a_{nn}>\frac{n-2}{(n-1)^2}$, we have $d>\frac{n-2}{(n-1)^2}$ as before. Thus we have

$$\begin{aligned} \operatorname{per}C(n,\ n) &= \left(\frac{n-2}{n-1}\right)^2 (n-3)!\,d^{n-3} \\ &= (n-2)!\,\frac{n-2}{(n-1)^2}\,d^{n-3} \\ &> (n-2)!\left(\frac{n-2}{(n-1)^2}\right)^{n-2} = \operatorname{per} T_n, \end{aligned}$$

contradicting Lemma 1 again. Hence

$a_{ij}\leqq\frac{n-2}{(n-1)^2}$ for all i, $j: 2\leqq i$, $j\leqq n$, which imply $a_{ij}=\frac{n-2}{(n-1)^2}$ for all i, j : $2\leqq i$, $j\leqq n$, as before. Thus $A=T_n$, completing the proof.

In [6], several proofs of the van der Waerden's conjecture:

If $X\in\Omega_n$ and $X\neq J_n$, then $\operatorname{per}X>\operatorname{per}J_n$, which has recently been solved by Egoryčev [1], are listed. A slight modification of the Egoryčev's proof of the van der Waerden's conjecture can be made to produce another proof which is different from those in [6], as follows.

If $A=[a_{ij}]\in\Omega_n$ is a minimizing matrix on Ω_n, then, as before, it must be that $a_{ij}\leqq\frac{1}{n}$ for all i, $j=1,\dots,n$, which implies that $A=J_n$.

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