HONAM MATHEMATICAL JOURNAL
Volume 7, Number 1, August, 1985

On Some Matrix Transformations Involving Prime Numbers*

by

V.K. Srinivasan

> *Abstract* <

The object of this note is to discuss the relationship between some matrix transformations that naturally occur involving prime numbers in the theory of Summability.

The Theory of Summability is replete with matrix transformations which are regular. Regular matrix transformations are those that not only preserve convergent sequences, but also preserve their limits. Among such transformations two of the popular transformations are the Norlund Means (N, q_n) and the Riesz Means $(\bar{N}, q_n)$. It is quite natural to consider the situation in which the sequence $\{q_n\}$ is replaced by the sequence of prime numbers. In this note we discuss two such transformations and their relationship with the $(C, 1)$ method.

Given a sequence $\{q_n\}$ of non-negative real numbers with $q_1>0$ the Norlund Means associated with $\{q_n\}$ denoted by (N, q_n) is defined in the following way. For any sequence $\{s_n\}$ the (N, q_n) transform of $\{s_n\}$ is defined as the sequence $\{t_n\}$ where

$$t_m=\frac{\sum_{n=1}^{m} q_{m-n+1}s_n}{Q_m} \qquad \cdots\cdots (1)$$

where $Q_m=q_1+q_2+\cdots\cdots+q_m$. It is well known that (N, q_n) is regular if and only if

$$\lim_{n\to\infty} (q_n/Q_n)=0. \qquad \cdots\cdots (2)$$

The corresponding Riesz Means $(\bar{N}, q_n)$ is the transformation for which the image sequence $\{t_n\}$ is defined by the rule

$$t_m=\frac{\sum_{r=1}^{m} q_r s_r}{Q_m} \qquad \cdots\cdots (3)$$

* Dedicated to Professor M. Venkataraman for his Sixtieth Birthday

It is well known that $(\bar{N}, q_n)$ is regular if and onlyif

$$\lim_{n\to\infty} (Q_n)=\infty \cdots\cdots (4)$$

Both the above transformations are particular cases of sequence to sequence transformations involving infinite matrices. Given an infinite matrix $A=(a_{n,k})$ $k=1,2,\cdots\cdots$, $n=1,2,\cdots\cdots$ of complex numbers for any sequence $\{s_n\}$ we define the A-transform to be the sequence $\{t_m\}$, where t_m is defined by

$$t_m=\sum_{k=1}^{\infty} a_{m,k}\, s_k \cdots\cdots (5)$$

If for each m the series in (5) converges and if

$$\lim_{m\to\infty} (t_m)=L \text{ exists} \cdots\cdots (6)$$

then the number L is called the A-limit of the sequence $\{s_n\}$. An infinite matrix A is said to be regular if for each convergent sequence $\{s_n\}$ the limit of $\{s_n\}$ coincides with the A-limit of $\{s_n\}$. The necessary and sufficient conditions for an infinite matrix to be regular are given in Hardy [1].

In this note we are primarily interested in the two transformations (N, p_n) and $(\bar{N}, p_n)$ where $\{p_n\}$ is the sequence of prime numbers with $p_1=2$, $p_2=3$, $p_3=5,\cdots\cdots$ It is easy to see on account of (4) that $(\bar{N}, p_n)$ is regular. However to see that (N, p_n) is regular we need to use some number theory results.

A fairly elementary result in number theory states that if $\{p_n\}$ denotes the n-th prime number, then there exist two positive constants C and D such that

$$C_n \log n\leq p_n\leq D_n \log n,\quad n\geq 2 \cdots\cdots (7)$$

A proof of result (7) is given in Niven and Zuckerman [2].

Theorem 1. Let $\{p_n\}$ be the sequence of primes. Let $P_n=p_1+p_2+\cdots\cdots+p_n$. Then

(1) $\lim_{n\to\infty} (p_n/P_n)=0$

(2) $np_n/P_n\leq K$ for all n, where K is a constant.

(3) (N, p_n) is a regular Norlund Mean.

Proof. (1) In order to prove (1) it suffices to prove that

$$\lim_{n\to\infty} (P_n/p_n)=\infty \cdots\cdots (8)$$

Consider

$$P_n/p_n = \frac{\sum_{r=1}^{n} p_r}{p_n} \quad \cdots\cdots (9)$$

$$\geq \frac{\sum_{r=[n/2]}^{n} p_r}{p_n} \quad \cdots\cdots (10)$$

where in (10) $[n/2]$ is the integral part of $n/2$. Hence

$$P_n/p_n \geq \frac{C \sum_{r=[n/2]}^{n} r \log r}{Dn \log n}$$

$$\geq \frac{(C/D) \sum_{r=[n/2]}^{n} r \log r}{n \log n}$$

$$\geq \frac{(C/D)\ (n-[n/2]) \cdot [n/2] \log [n/2]}{n \log n} \quad \cdots\cdots (11)$$

From (11) it is easily seen that (8) holds and hence (1) follows.

(2) From (11) it is seen that

$$P_n/np_n \geq \frac{(C/D)\ (n-[n/2])\ [n/2] \log [n/2]}{n^2 \log n} \quad \cdots\cdots (12)$$

Since the right hand side of the above relation (12) approaches a limit greater than 0, it follows that relation (2) holds in the theorem.

(3) of the theorem now follows step (1) using (1) of the present theorem. This completes the proof.

If S and T are two regular transformations we say that S is stronger than T (in symbols $S \supseteq T$) if for every T-limitable sequence $\{s_n\}$, the sequence is also S-limitable to the same limit.

A necessary and sufficient condition for a matrix $A=(a_{m,n})$ to be stronger than the $(C,1)$ mean is that there exists a constant K with

$$\sum_{n=1}^{\infty} n\,|a_{m,n} - a_{m,n+1}| \leq K \quad \cdots\cdots (13)$$

We need the following

Lemma 1. $(\bar{N}, p_n)$ is stronger than $(C, 1)$.

Proof. For $(\bar{N}, p_n)$ we consider the inequality (13). Here

$$\sum_{n=1}^{\infty} n|a_{m,n}-a_{m,n+1}|=\sum_{n=1}^{m-1} n|p_n/P_m-p_{n+1}/P_m|+mp_m/P_m$$

$$\leq(m/P_m)\left\{\sum_{n=1}^{m-1}(p_{n+1}-p_n)+p_m\right\}$$

$$\leq(m/P_m)\ (2p_m-p_1) \quad \cdots\cdots (14)$$

From (14) it is quite easy to see using (2) of Theorem 1 that the right hand side of inequality (14) is further $\leq L$, where L is an absolute constant. The result now follows from (13).

If $\{q_n\}$ and $\{r_n\}$ are two positive sequences such that $\sum q_n=\infty$ and $\sum r_n=\infty$ and if $q_{n+1}/q_n\leq r_{n+1}/r_n$ then $(\bar{N}, q_n)$ is stronger than $(\bar{N}, r_n)$. This quoted result is proved in Hardy [1] (page 58, Theorem 14). By taking $q_n=1$ for all n and $r_n=p_n$ (the sequence of primes) we now obtain.

Lemma 2. The $(C, 1)$ mean is stronger than $(\bar{N}, p_n)$.

Combining Lemma 1 and Lemma 2 we obtain the following

Theorem 2. The Riesz mean $(\bar{N}, p_n)$ is equivalent to the $(C, 1)$ mean.

It is known that the Norlund method (N, q_n) is stronger than $(C, 1)$ when $\{q_n\}$ is an increasing sequence. Since $\{p_n\}$ is an increasing sequence $(N, p_n)\supseteq(C, 1)$. If follows that $(N, p_n)\supseteq(\bar{N}, p_n)$.

We now state without proofs some of the properties of (N, p_n) and $(\bar{N}, p_n)$. These proofs follow by using well known results from Summability theory.

Theorem 3. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, converging respectively to A and B then the Cauchy product series $\sum_{n=1}^{\infty} c_n$ converges to $A.B$, when summed by (N, p_n) or by $(\bar{N}, p_n)$.

Theorem 4. Let $t_m=x_m+t\left\{\dfrac{p_1x_1+p_2x_2+\cdots\cdots+p_mx_m}{P_m}\right\}$. If $|t|<1$, then $\{t_m\}$ converges to s if and only if $\{s_m\}$ converges to s.

Theorem 5. Let $t_m=x_m+t\left\{\dfrac{p_1x_m+p_2x_{m-1}+\cdots\cdots+p_mx_1}{P_m}\right\}$ If $|t|<1$, then the sequence $\{t_m\}$ converges if and only if $\{x_m\}$ converges to s.

Both (N, p_n) and $(\bar{N}, p_n)$ sum divergent sequences of zeros and ones.

For $(\bar{N}, p_n)$, $a_n=O(p_n/P_n)$ is a Tauberian condition. For the definition of strongly regular transformations we refer to Petersen [3]. Both the above transformations are strongly regular.

The author would like to thank the referee for correcting some of the mistakes in the earlier draft of this note.

References

1. G.H. Hardy., Divergent Series, Oxford University Press, London, 1949.
2. I. Niven and H. Zuckerman., An Introduction to Theory of Numbers, John Wiley and Sons Inc., New York, 1967.
3. G.M. Petersen., Regular matrix transformations, Mcgraw Hill publishing Company, England, 1966.
4. A. Peyerimhoff., Lectures on Summability, Lecture Notes in Mathematics, 107, Springer-Verlag, New York, 1969.
5. R.E. Powell and S.M. Shah., Summability Theory and Its Applications, Van Nostrand and Reinhold Company, London, 1972.

Department of Mathematics, University of Texas At El Paso,
El Paso, Texas, 79968, U.S.A.