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# A Note on the Ring $I_{\mathrm{k}/k}$ of Integers

by

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Throughout this paper we assume that

$Z$=the ring of all integers, $Q$=the rational field
$C$=the complex field,

Moreover, we assume that K and $k$ are number fields such that $[\mathrm{k}:k]=n$ and $[\mathrm{k}:Q]=m$ and that $I_k$ is the ring of integers of $k$. The purpose of this paper is to define the ring $I_{\mathrm{k}/k}$ of integers of $\mathrm{k}/k$(Definition 1) and to prove that ① $I_{\mathrm{k}}=I_{\mathrm{k}/k}$(Theorem 3) ② $I_{\mathrm{k}/k}$ is a Dedekind integral domain(corollary 6). As is well known $I_k$ is a Dedekind integral domain and thus $I_k$ has the unique factorization property with respect to prime ideals. (※)

Let k be ane extension field of $k$ with $[\mathrm{k}:k]=n$.(Note that the complex filed $C\supset \mathrm{K}\supset k\supset Q$ and $(\mathrm{k}:Q)=[\mathrm{k}:k]\,[k:Q]=mn$).

**Definition 1.** If $\alpha\in\mathrm{k}$ satisfies

$$\alpha^r+\alpha_1\alpha^{r-1}+\cdots+\alpha_{r-1}+\alpha_r=0 \quad (\alpha_i\in I_k,\ i=1,\cdots r)$$

then $\alpha$ is called an algebraic integer of k over $k$. $I_{\mathrm{k}/k}$ denotes the set of all algebraic integers of k over $k$.

**Definition 2.** Let $\mathrm{k}=k(\theta)$ and let $f(x)=X^n+a_1X^{n-1}+\cdots+a_{n-1}X+a_n(a_i\in\mathrm{k}$ and $i=1,n)$ be a minimal polynomial of $\theta$. Let us denote the conjugate elements of $\theta$ by

$$\theta=\theta^{(1)},\ \theta^{(2)},\cdots,\theta^{(n)},$$

then $f(\theta^{(i)})=0$ for $i=1,\cdots,n$. We put $\mathrm{K}^{(i)}=k(\theta^{(i)})$ for $i=1,\cdots,n$. Each $\alpha\in\mathrm{K}$ has a unique representation

$$\alpha=a_0+a_1\theta+\cdots+a_{n-1}\theta^{n-1}\ (a_i\in k,\ i=0,\cdots,\ n-1).$$

We define the algebraic conjugates of $\alpha$ by

$$\alpha=\alpha^{(1)}=d_0+d_1\theta^{(1)}+\cdots+\alpha_{n-1}\ \theta^{(1)n-1}$$
$$\alpha^{(2)}=\alpha_0+\alpha_1\theta^{(2)}+\cdots+\alpha_{n-1}\theta^{(2)n-1}$$
$$\vdots$$
$$\alpha^{(n)}=\alpha_0+\alpha_1\theta^{(n)}+\cdots+\alpha_{n-1}\theta^{(n)n-1}.$$

(Note that if $\alpha\in k$ then $\alpha^{1}=\cdots=\alpha^{(n)}=\alpha$.). The norm $N_{k/k}\ \alpha$ and the trace $T_{k/k}\ \alpha$ of $\alpha$ are defined by

$$N_{k/k}\ \alpha=\alpha^{(1)}\cdots\alpha^{(n)},\quad T_{k/k}\ \alpha=\alpha^{(1)}+\cdots+\alpha^{(n)},$$

respectivelly.

**Theorem 1.** We have the following properties about $I_{k/k}$.

(i) $I_{k/k}$ is an integral domain.
(ii) $\forall\alpha\in K\ \exists m(\neq 0)\in Z$ such that $m\alpha\in I_{k/k}$.
(iii) $I_{k/k}$ is integrally closed, *i,e.*, if $\alpha\in K$ satisfies
$\alpha^r+\alpha_1\alpha^{r-1}+\cdots+\alpha_{r-1}\alpha+\alpha_r=0$ ($\alpha_i\in I_{k/k}$, $i=1,\cdots,r$) then $\alpha\in I_{k/k}$.
(iv) $I_{k/k}\cap k=I_k$
(v) For $\alpha\in I_{k/k}$ $N_{k/k}\alpha$, $T_{k/k}\ \alpha\in I_k$

Furthermore, the algehraic conjugates $\alpha=\alpha^{(1)},\cdots\alpha^{(n)}$ $\alpha$ of $\alpha$ are also algebriac integers over $k$, *i.e*, $\alpha^{(1)}$ is an algebraic integer of $K^{(i)}$ over $k$ for $i=1,\cdots n(\alpha^{(i)}\in I_{k^{(i)}/k})$.

**Proof (i)**: We shall prove that $\lambda\in I_{k/k}\leftrightarrow\exists\eta_1,\cdots,\eta_n\in K$ such that ① $\eta_i\eta_j\in I_k\eta_1+\cdots+I_k\eta_n$ $(i,j=1,\cdots n)$

② $\lambda\in I_k\eta_1+\cdots+I_k\eta_n$

Suppose $\lambda\in I_{k/k}$, then there exist $\alpha_i\in I_k$ such that

$$\lambda^r+\alpha_1\ \lambda^{r-1}+\cdots+\alpha_{r-1}\ \lambda+\alpha_r=0.$$

Thus, put $\lambda^{r-1}=\eta_1,\cdots,\lambda=\eta_{r-1}$ and $1=\eta_r$, then

$$\eta_i\eta_j=\lambda^{2r-(i+j)}=\begin{cases}\eta_{i+j-r} & \text{if } i+i>r\\ b_1\eta_1+\cdots b_m\eta_r & \text{if } i+j\leqq r\end{cases}$$

where $b_1$ $(i=1,\cdots r)\in I_k$. That is, $\eta_i\eta_j\in I_k\ \eta_1+\cdots+I_k\eta_r$, Next,

$$\lambda={}_{r-1}\Longrightarrow\lambda\in I_k\eta_1+\cdots+I_k\eta_n.$$

Conversely, suppose $\lambda\in I_k\eta_1+\cdots+I_k\eta_n$, By our assumptions ① and ②

$$\lambda\eta_i=\sum_{i,j}^{n}c_{ij}\eta_j\ \ (c_{ij}\in I_k).$$

Define the $N\times N$-matrix $c=(\delta_{ij}\lambda-c_{ij})_{i,j}=1\cdots N$

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j.\end{cases}$$

In this case:

$$C\begin{pmatrix}\eta_1\\ \vdots\\ \eta_N\end{pmatrix}=\begin{pmatrix}\lambda-C_1 & -C_{12}\cdots & -C_{1N}\\ -C_{21} & \lambda-C_{22}\cdots & -C_{2N}\\ \vdots & \vdots & \\ -C_{N1} & -C_{N2}\lambda\cdots\lambda-C_{NN}\end{pmatrix}\begin{pmatrix}\eta_1\\ \vdots\\ \eta_N\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

If det $C\neq 0$ then there exists the inverse $C^{-1}$ and we have

$$\begin{pmatrix}\eta_1\\ \vdots\\ \eta_N\end{pmatrix}=C^{-1}C\begin{pmatrix}\eta^1\\ \vdots\\ \eta_N\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

In consequence, $\eta_1=\cdots=\eta_N=0$ and $\lambda=0$. If det $C=0$ then $\lambda$ is a root of the following equation

$$\begin{vmatrix}\lambda-C_{11},\cdots-C_{12}\cdots-C_{1N}\\ \vdots \quad \ddots \quad \vdots\\ -C_{N1}\cdots\lambda-C_{NN}\ \lambda-C_{NN},\end{vmatrix}=0 \quad (C_{ij}\in I_k)$$

and thus $\lambda\in I_{k/k}$.

We shall prove (i) by using the above result. Take $\alpha,\beta\in I_{k/k}$, then

$$\alpha\in I_k\eta_1+\cdots+I_k\eta_N,\ \ \eta_i\eta_j\in I_k\eta_1+\cdots I_k\eta_N$$
$$\beta\in I_k\zeta_1+\cdots+I_k\zeta_M,\ \ \zeta_i\zeta_j\in I_k\zeta_1+\cdots+I_k\zeta_M,$$

where $\eta_i$, $\zeta_j\in$K for $i=1,\cdots,$ $N$ and $j=1,\cdots,$ M. Since

$$\eta_i\eta_j\in I_k\eta_1+\cdots+I_k\eta_N\ \ I_k\eta_1+\cdots+I_k\eta_N+I_k\zeta_1+\cdots+I_k\zeta_M+I_k\eta_1\zeta_1+\cdots+I_k\eta_N\zeta_M$$
$$\zeta_i\zeta_j\in I_k\zeta_1+\cdots+I_k\zeta_M\subset I_k\eta_1+\cdots+I_k\zeta_1+\cdots+I_k\eta_1\zeta_1+\cdots+I_k\eta_N\zeta_M$$
$$(\eta_i\zeta_j)(\eta_k\zeta_l)=(\eta_i\eta_k)(\zeta_j\zeta_l)\in I_k\eta_1\zeta_1+\cdots+I_k\eta_N\zeta_M$$
$$CI_k\eta_1+\cdots+I_k\zeta_1+\cdots+I_k\eta_N\zeta_M$$

and

$$\alpha\pm\beta,\alpha\beta\in I_k\eta_1+\cdots+I_k\eta_N+I_k\zeta_1+\cdots+I_k\zeta_M+I_k\eta_1+\cdots+I_k\eta_N\zeta_M$$

it follows that $\alpha\pm\beta,\alpha\beta\in I_{k/k}$. Thus $I_{k/k}$ is a commutative ring. It is clear that $I\in I_{k/k}$, and thus $I_{k/k}$ is an integral domain.

(ii): Let $X^r+\alpha_1X^{r-1}+\cdots+\alpha_{r-1}X+\alpha_r(\alpha_i\in k,\ i=1,\cdots,\ r,\ r\leq n)$ be a minimal polynomial of $\alpha\in$K orer $k$. As in [2] there exist non-zero rational integers $m_1,\cdots m_r$ such that

$$m_1\alpha_1,\cdots,m_r\alpha_r\in I_k$$

Put $b_0=m_1m_r$ ($\in Z$) then $\alpha$ is a root of

$$b_0X^r+\beta_1X^{r-1}+\cdots\beta_{r-1}X+\beta_r=0.\ (\beta_i\in I_k,\ i=1,\cdots r),$$

where $\beta_1=b_0\alpha_1,\cdots$, and $\beta_r=b_0\alpha_r$, $i,e.$,

$$b_0\alpha_r+\beta_i\alpha^{r-1}+\cdots+\beta_{r-1}\alpha+\beta_r=0.$$

Thus, we have

$$(b_0\alpha)^r+\beta_1b_0(b_0\alpha)^{r-1}+\cdots+\beta_{r-1}\ b_0^{r-2}\ (b_0\alpha)+b_0^{r-1}\ \beta_r=0.$$

If we put $\beta_1b_0=\gamma_1,\cdots,\ \beta_{r-1}b_0^{r-2}=\gamma_{r-1},\ b_0^{r-1}\beta_r=\gamma_r$ then $b_0\alpha$ is a root of

$$X^r+\gamma_1X^{r-1}+\cdots+\gamma_{r-1}X+\gamma_r=0,\ (\gamma_i\in I_k,\ i=1,\cdots,\gamma),$$

and thus $b_0\alpha\in I_k$.

(iii) Suppose $\alpha\in$K satisfies

$$\alpha^r+\alpha_1\alpha^{r-1}+\cdots+\alpha_{r-1}\alpha+\alpha_r=0\ (\alpha_i\in I_{k/k},\ i=1,\cdots,\gamma.)$$

Since $\alpha_i\in I_{k/k}$ we have some positive rational integer $N_1$ such that

$$\alpha_1^{N_i}=-(\beta_i\alpha_i^{N_i-1}+\cdots+\beta_{N_{i-1}}\ \alpha_i-\beta_{N_i})\ (\beta_i\in I_k,\ i=1,\cdots N_i)$$

Put $\alpha_1^{N_i-1}=\eta_{ij},\cdots,\ \alpha_i=\eta_{iN_{i-1}}$ and $1=\eta_{iN_j}$, then

$$\eta_{ij}\eta_{ik}\in I_k\eta_{i1}+\cdots+I_k\eta_{iN_j}$$
$$\alpha_i\in I_k\eta_{i1}+\cdots+I_k\eta_{iN_i},\ (i=1,\cdots\gamma).$$

Let us put

$$\zeta_{lj1\cdots jr}=\alpha^l\eta_{1j1}\cdots\eta_{rjr}\ (1=0,1,\cdots,\ r-1,{}_{ji}=1,\cdots,N_i,\ i=1,\cdots r)$$

then it follows that

$$\zeta_{lj1\cdots jr},\ \zeta_{l'},{}_{j1'}\cdots{}_{jr'},\in\sum I_k\zeta_{lj1\cdots jr}$$
$$\alpha\in\sum I_k\ \zeta_{lj1\cdots jr}.$$

That is, by(i) $\alpha\in I_{k/k}$.

(iv) Note that

$I_{k/k}\cap k$=the set of all algebraic integers over $k$ in $k$.

Since $I_k$ is integrally closed in $k$[As in [2]] it follows that $I_{k/k}\cap k=I_k$.

(v): Let $\alpha=\alpha_0+\alpha_1\theta+\cdots+\alpha_{n-1}\ \theta^{n-1}\in I_{k/k}$(see Definition 2), and let $X^r+\beta_1X^{r-1}+\cdots+\beta_{r-1}X+\beta_r(\beta_i\in I_k,\ i=1,\cdots\gamma)$ be a minimal polynomial of $\alpha$ over $k$. Then

$$\alpha^r+\beta^{r-1}+\cdots+\beta_{r-1}\alpha+\beta_r=0.$$

The algebraic conjrgate $\alpha^{(i)}=\alpha_0+\alpha_1\theta^{(i)}+\cdots+\alpha_{n-1}\theta^{(i)n-1}$ $(i=1,\ \cdots,\ n)$ of $\alpha$ saticfies the above equation, *i,e,*

$$\alpha^{(i)r}+\beta_1\ \alpha^{(i)r-1}+\cdots+\beta_{r-1}\alpha^{(i)}+\beta_r=0.$$

Thus $\alpha^{(i)}$ is analgebaic integer over $k(\alpha^{(i)}\in I_k{}^{(i)}/k)$.

On the other hand, $N_{k/k}\alpha=\alpha^{(1)}\cdots\alpha^{(n)}$ and $T_{k/k}$ $(\alpha^{(1)}+\cdots+\alpha^{(n)}$ are symmetric functions of $\theta^{(1)},\cdots,$ and $\theta^{(n)}$, and thus

$$N_{k/k}\ \alpha,\ T_{k/k}\in k.$$

Let $L$ be an extension field of $K^{(1)},\cdots,$ and $K^{(n)}$ suchthat $L\subset C$(the cemplex field). Then, $\alpha^1,\cdots,\ \alpha^{(n)}\in I_{L/k}$, and also $N_{k/k}\ \alpha,\ T_{k/k}\ \alpha\in I_{L/k}$ by (i). Thus,

$$N_{k/k}\alpha,\ T_{k/k}\ \alpha\in I_{L/k}\cap k=I_k$$

by (iv). ///.

**Theorem 2.** When $[K:\ k]=n$ we have

$$I_{k/k}=I_kw_1\oplus\cdots\oplus I_kw_n \text{ (direct sum)},$$

where $w_1\cdots,$ and $w_n$ are elements of K. (We say that $\{w_1,\cdots,w_n\}$ is a basis of $I_{k/k}$.)

**Proof.** As in Definition 2, we put $K=k(\theta)=k\oplus k\theta\oplus\cdots\oplus k\theta^{n-1}(\theta\in K)$. By (ii) of Theorem 1. there exists a non-zero rational integer $m$ such that $m\theta\in I_{k/k}$. Thus, we may assume that $\theta\in I_{k/k}$. Then we have a unique expression of $\alpha\in K$ such that

$$\alpha=\alpha_0+\alpha_1\theta+\cdots+,\ \alpha_{n-1}\theta^{n-1}\ (\alpha_i\in k,\ i=0,1,\cdots,\ n-1).$$

Let

$$\alpha=\alpha^{(1)}=\alpha_0+\alpha_1\theta^{(1)}+\cdots+\alpha_{n-1}\theta^{(1)n-1}$$
$$\vdots$$
$$\alpha^{(n)}=\alpha_0+\alpha_1\theta^{(n)}+\cdots+\alpha_{n-1}\theta^{(n)n-1}$$

be the algebraic conjngate of $\alpha$. We put

$$\triangle=\begin{vmatrix} 1 & \theta^{(1)} & \cdots & \theta^{(1)n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \theta^{(n)} & \cdots & \theta^{(n)n-1} \end{vmatrix}=\underset{i<j}{\pi}(\theta^{(i)}-\theta^{(j)})$$

and

$$\triangle^{(i+1)}=\begin{vmatrix} 1 & \theta^{(1)} & \cdots & \theta^{(1)i-1} & \alpha^{(1)} & \theta^{(1)i+1} & \cdots & \theta^{(1)n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \theta^{(n)} & \cdots & \theta^{(n)i-1} & \alpha^{(n)} & \theta^{(n)i+1} & \cdots & \theta^{(n)n-1}, \end{vmatrix}$$

then $\theta^{(i)}\neq\theta^{(j)}$ $(i\neq j)\Longrightarrow\triangle\neq 0$ and $\alpha_i=\triangle^{(i+1)}/\triangle$. Since $\theta\in I_{k/k}$ and $\triangle^2$ is a symmetric function of $\theta=\theta^{(1)}\cdots$and $\theta^{(n)}$ $\triangle^2\in I_k$. In

$$\alpha_i=\frac{\triangle\triangle^{(i+1)}}{\triangle^2}$$

$\triangle^2\in I_k$ and $\alpha_i$ $\triangle^2=\triangle\triangle^{(i+1)}\in k(\alpha_i\in k)$. If $\alpha\in I_{k/k}$ $\alpha_i\in I_k$.
Thus, $\alpha\in I_{k/k}\Longrightarrow\triangle^2$, $\alpha_i\triangle^2=\triangle\triangle^{(i+1)}\in I_k$. Therefore, when $\alpha\in I_{k/k}$ we have the following):

$$\begin{aligned}\alpha&=\alpha_0+\alpha_1\theta+\cdots+\alpha_{n-1}\theta^{n-1}(\theta\in I_{k/k},\ \alpha_i\in I_k,\ i=0,\cdots n-1)\\ &=\frac{\triangle\triangle^{(1)}}{\triangle^2}+\frac{\triangle\triangle^{(2)}}{\triangle^2}\theta+\cdots+\frac{\triangle\triangle^{(n)}}{\triangle^2}\theta^{n-1}\\ &=\triangle\triangle^{(1)}\frac{1}{\triangle^2}+\triangle\triangle^{(2)}\frac{\theta}{\triangle^2}+\cdots+\triangle\triangle^{(n)}\frac{\theta^{n-1}}{\triangle^2}\\ &\in I_k\frac{1}{\triangle^2}\oplus I_k\frac{\theta}{\triangle^2}\oplus\cdots\oplus I_k\frac{\theta^{n-1}}{\triangle^2}\end{aligned}$$

In consequence

$$I_{k/k}\subseteq I_k\frac{1}{\triangle^2}\oplus I_k\frac{\theta}{\triangle^2}\oplus\cdots\oplus I_k\frac{\theta^{n-1}}{\triangle^2}\ \text{(direct sum)}$$

Where the right hand side is a free $I_k$-module with a basis $\left\{\frac{1}{\triangle^2},\frac{\theta}{\triangle^2},\cdots,\frac{\theta^{n-1}}{\triangle^2}\right\}$. Thus, $I_{k/k}$ is a $I_k$-submodule of a free $I_k$-module. It follows that $I_{k/k}$ has a basis $\{w_1,\cdots w_n\}$ such that

$$I_{k/k}=I_kw_1\oplus\cdots\oplus I_kw_n$$

([10]). ///

**Theorem 3.** $I_{k/k}=I_k$.

**Proof.** It is clear that $I_k\subset I_{k/k}$ because that $Z\subset I_k$ Suppose $[K:k]=n$, $[k,Q]=m$ and

$$I_{k/k}=I_k w_1\oplus\cdots\oplus I_k w_n \quad (w_i\in I_{k/k},\ i=1,\cdots,n)$$
$$I_k=Z\zeta_1\oplus\cdots\oplus Z\zeta_m \quad (\zeta_j\in I_k,\ j=1,\cdots,m).$$

Then, we have

$$I_{k/k}=Z\zeta w_1\oplus\cdots\oplus Z\zeta_m w_1\oplus Z\zeta w^2\oplus\cdots\oplus Z\zeta_m w_n,$$

which is a direct sum because that $\zeta_1 w_1,\cdots,$ and $\zeta_m w_n$ are linearlly independent over $Z$. Furthermore, since $I_k\subset I_{k/k}$

$$\zeta_1 w_1,\cdots,\ \zeta_m w_n\in I_{k/k}.$$

Next, we want to prove that $w_i\ (i=1,\cdots,n)\in I_k$. Let

$$g(X/_k)=X^n+\alpha_1 X^{n-1}+\cdots+\alpha_{n-1}X+\alpha_n(\alpha_i\in I_k,\ i=1,\cdots,n)$$

be a minimal polynonial of $w_i\in I_{k/k}$.

Let us put

$k^{(1)}=k,\cdots k^{(m)}$ as the conjugate fields of $k$ over $Q$

$$g(X/_k{}^{(j)})=X^n+\alpha_1{}^{(j)}X^{n-1}+\cdots+\alpha_{n-1}^{(j)}X+\alpha_n{}^{(j)}(\alpha_i{}^{(j)}\in k^{(j)}\ i=1,\cdots,n).$$

Then a minimal polynomial of $w_i$ over $Q$ is

$$G(X)=g(X/_k{}^{(1)}),\ g(X/_k{}^{(2)})\cdots g(X/_k{}^{(m)}).$$

Take a number field $L$ such that $L\supset K^{(1)},\cdots K^{(n)}$. Then $\alpha_i{}^{(j)}\in I_L$ for $i,j=1\cdots n$ and each coefficient of $G(X)$ is in $Q$.

Since $Q\cap I_L=Z$ as in [2] we have each coefficient of

$$G(X)\in Z.$$

It is clear that $G(w_i)=0$, and thus $wi$ is an element of $I_k$.
Therefore we have $\zeta_i w_j\in I_k$ for $i=1,\cdots,m$ and $j=1,\cdots,n$, because that $I_k\subset I_k$. In consequence, we have the following:

$$I_k=Z\zeta_1 w_1\oplus Z\zeta_2 w_1\oplus\cdots\oplus Z\zeta_m w_n)$$

$i,e.,\ I_k=I_{k/k}.$ ///

Corollary, $I_{k/k}$ is a Dedekind integral domain.

**Proof**. By (※) above, since $I_k$ is a Dedekind integral domain so is $I_{k/k}$ by Theorem 3. We have one other proof as follows.

We havetwo steps to do this.

**Step I.** Let us assume that

$$I_{k/k}=I_k w_1\oplus\cdots\oplus I_k w_n$$
$$I_k=Z\zeta_1\oplus\cdots\oplus Z\zeta_m.$$

Then, as before,

$$I_k=Z\zeta_1 w_1\oplus\cdots\oplus Z\zeta_m w_1\oplus\cdots\oplus Z\zeta_m w_n.$$

Let $\sqcap$ be on ideal of $I_{k/k}$. Since $\sqcap$ is a $I_k$-submodule of the free $I_k$-module $I_{k/k}$ it has a basis $\{\alpha_1,\cdots,\alpha_n\}$ such that

$$\sqcap=I_k\alpha_1\ \oplus\cdots\oplus I_k\alpha_n.$$

If we pus

$$\sqcap^1=Z\zeta_1\alpha_1\oplus\cdots\oplus Z\zeta_m\alpha_1\oplus Z\zeta_1\alpha_2\oplus\cdots\oplus Z_1\zeta_m\alpha_n,$$

then $\sqcap^1$ is an idal of $I_k$ generated {by $\zeta_1\alpha_1,\cdots,\zeta_m\alpha_n\}$. There exists a maximum matrix $A=(a_{ij})$ $(a_{ij}\in Z)$ such that

$$\begin{pmatrix}\zeta_1\alpha_1\\ \vdots\\ \zeta_m\alpha_1\\ \zeta_1\alpha_2\\ \vdots\\ \zeta_m\alpha_n\end{pmatrix}=A\begin{pmatrix}\zeta_1 w_1\\ \vdots\\ \zeta_m w_1\\ \zeta_m w_2\\ \vdots\\ \zeta_m w_n\end{pmatrix}$$

By a choice of $\{\zeta_1\alpha_1,\cdots,\ \zeta_m\alpha_n\}$ and $\{\zeta_1 w_1,\cdots\zeta_m w_n\}$ A can be denoted by

$$\det\ A=\begin{vmatrix}e_1 & & 0\\ & \ddots & \\ 0 & & e_{mn}\end{vmatrix}\ (e_i\in Z,\ \ i=1,2,\cdots,\ mn).$$

Therefore

$$N\sqcap^1=|\det A|=|e_1\cdots e_{mn}|,$$

which called the norm of $\sqcap^1$. Thus, each element of $I_k/\sqcap^1$ is uniquely represented by

$$a_1\zeta_1 w_1+\cdots+a_{mn}\zeta_m w_n,$$

where $0\leqq a_j<1e_j|$ for $j=1,2,\cdots,\ mn$. The number of elements in $T_k/\sqcup^1$ is just $1e_1\cdots e_{mn}$ $1=|\det A|$. Put

$$\beta_1 = \sum_{i=1}^{m} {}^{r} a_i \zeta_i, \ \beta_2 \sum_{i=1}^{n} a_{m+i} \ \zeta_i, \cdots, \ \beta_n = \sum_{i=1}^{m} a_{mn-m+i} \zeta_i$$

then each element of $I_{k/k}/\sqcap$ is represented by a unique form:

$$\beta_1 w_1 + \cdots + \beta_n w_n.$$

Since $I_k/\sqcap^1$ is a finite set, so is $I_{k/k}/\sqcap$.

**Step II.** (1) $I_{k/k}$ is integrally closed (iii) of Theorem 1.

(2) $I_{k/k}$ is a Noetherian ring.

Suppose a series of ideals $\sqcap_1 \subsetneqq \sqcap_2 \subsetneqq$ in $I_{k/k}$. Then there corresponds a series of ideals $\sqcap_1' \subsetneqq \sqcap_2' \subsetneqq$ in $I_k$.

By (※) above there exists a positive rational integer $l$ such that $\sqcap_l' = I_k$. Thus $\sqcap_l = I_{k/k}$ and $I_{k/k}$ is Noetherian.

(3) For a prime ideal $\sqcap$ of $I_{k/k}$ the integral domain $I_{k/k}/\sqcap$ is a dinite set as above. This implies that $I_{k/k}/\sqcap$ is a field. Hence $\sqcap$ is a maximal ideal of $I_{k/k}$.

Sumning up ths above things, $I_{k/k}$ is a Dedekind integral domain. ///

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