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On the Cohomology Spectral Sequences

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1. Introduction

The Cohomology Spectral Sequence is applied to the many respects of mathematics. For example, it plays an essential role in Topology ([8]. [9]. [11]), Sheaf Theory, Algebraic Geometry and Algebra ([3], [4], [7], [13]).

We can see the study of spectral sequence in the many researching papers; i.e., ([2], [3], [11], [15]), and we know that spectral sequence can be derived by means of exact couple; i.e., ([9], [13], [14]). This dissertation is a study on cohomology spectral sequence as a concept of the duality of spectral sequence concept defined in [13]. The fact that cohomology spectral sequence always exists when complex and its filtration are given, is verified in Theorem 3.4.. And again the fact that cohomology spectral sequence exists when bicomplex and first filtration are given, is verified in Theorem 4.3..
In particular, I tried to show each concrete elements of

$E_2^{p,q}$ and $E_3^{p,q}$ in Theorem 4.3.. For example, $E_2^{p,q}=L_2^{p,q}/M_2^{p,q}$, $E_3^{p,q}=L_3^{p,q}/M_3^{p,q}$,
where $L_2^{p,q}=\{a^{p,q}\in K^{p,q}\mid \delta'' a^{p,q}=0$ and $\exists\ a^{p+1,q-1}\in K^{p+1,q-1}$ such that $\delta' a^{p,q}=\delta'' a^{p+1,q-1}\}$
$M_2^{p,q}=\{\delta' b^{p-1,q}+\delta'' b^{p,q-1}\in K^{p,q}\mid \delta'' b^{p-1,q}=0,\ b^{p-1,q}\in K^{p-1,q},\ b^{p,q-1}\in K^{p,q-1}\}$

and so, when we want to inquire into the nature of cohomology spectral sequence, we need Künneth Formula, and the content of the §2 is about the revision of Künneth Formula suitable for our purpose. That is to say, in our case Künneth Formula develops and leads to the final stage as follows;

$$\sum_{p+q=n} H^p(X)\otimes H^q(Y) \rightarrowtail^{\alpha} H^n(X\otimes Y) \xrightarrow{\beta}\!\!\!\!\rightarrow \sum_{p+q=n+1} \mathrm{Tor}_1(H^*(X),\ H^*(Y)).$$

Theorem 4.5. is the result of the application of Künneth Formula. For example, when X and Y stand for complexes of Abelian groups with X^n a free group,

$E_2^{p,q} \cong H_I^p(X \otimes H_2^q(Y))$ may appear, where $E_2 = E_3 = \cdots = E_\infty$ is maintained.

When spectral sequence (E^r, d^r) is given and each E^r is a vector space over a field F, Here the possibility to get cohomology spectral sequence has been verified in example 3.5..

In this paper we promise that $\boldsymbol{Z}$ should represent the ring of integers.

2. Künneth Formulas

Let G and A be additive abelian groups and let $\boldsymbol{Z}$ be the ring of integers. The *torsion product* Tor(G, A) of G and A is an abelian group generated by

$$\{\langle g,m,a\rangle \mid m \in \boldsymbol{Z},\ g \in G,\ a \in A,\ gm = 0 = ma\},$$

where each $\langle g,m,a\rangle$ subjects to the relations

$$\left.\begin{array}{l} \langle g_1+g_2,\ m,a\rangle = \langle g_1,m,a\rangle + \langle g_2,m,a\rangle \text{ if } g_i m = 0 = ma\ (i=1,2) \\ \langle g,m,a_1+a_2\rangle = \langle g,m,a_1\rangle + \langle g,m,a_2\rangle \text{ if } gm = 0 = ma_i\ (i=1,2) \\ \langle g,mn,a\rangle = \langle gm,n,a\rangle \text{ if } gmn = 0 = na \\ \langle g,mn,a\rangle = \langle g,m,na\rangle \text{ if } gm = 0 = mna \end{array}\right\} \quad (※)_1$$

Proposition 2.1. Let A_1 and A_2 be abelian groups and let G and A be additive abelian groups.

(i) $gm = 0 = ma \Longrightarrow \langle o,m,a\rangle = 0 = \langle g,m,o\rangle$

(ii) if A is torsion-free, then Tor$(G,\ A) = 0$

(iii) Tor$(G,\ A) \cong$ Tor$(A,\ G)$

(iv) Tor$(G,\ A_1 \oplus A_2) \cong$ Tor$(G,\ A_1) \oplus$ Tor$(G,\ A_2)$.

Proof. (i) $\langle g+o,\ m,a\rangle = \langle g,m,a\rangle + \langle o,m,a\rangle \Longrightarrow \langle o,m,a\rangle = 0.$
Similarly,

$$\langle g,m,a+o\rangle = \langle g,m,a\rangle + \langle g,m,o\rangle \Longrightarrow \langle g,m,o\rangle = 0.$$

(ii) By our hypotheses, for each element $a(\neq 0) \in A$ there is no positive integer $m \in \boldsymbol{Z}$ such that $ma = 0$.

Hence Tor$(G,\ A)$ is generated by the elements $\langle g,m,o\rangle$ where $gm = 0$.
But by (i) $\langle g,m,o\rangle = 0$ and thus Tor$(G, A) = 0$.

(iii) There exists a homomorphism

$$\varphi;\ \text{Tor}(G,A) \longrightarrow \text{Tor}(A,G)$$

such that $\varphi(<g,m,a>)=<a,m,g>$.

Similarly, there exists a homomorphism

$$\psi\colon \operatorname{Tor}(A,G)\longrightarrow\operatorname{Tor}(G,A)$$

such that $\psi(<a,m,g>)=<g,m,a>$.

Since $\psi=\varphi^{-1}$, we see that ψ is an isomorphism.

Hence $\operatorname{Tor}(G,A)\cong\operatorname{Tor}(A,G)$.

(iv) There exists a group homomorphism φ: $\operatorname{Tor}(G,A_1\oplus A_2)\longrightarrow\operatorname{Tor}(G,A_1)\oplus\operatorname{Tor}(G,A_2)$ such that $\varphi(<g,m,(a_1,a_2)>)=(<g,m,a_1>,\ <g,m,a_2>)$.

Clearly there exist group homomorphisms

$$\psi_1\colon \operatorname{Tor}(G,A_1)\longrightarrow\operatorname{Tor}(G,A_1\oplus A_2),$$
$$\psi_2\colon \operatorname{Tor}(G,A_2)\longrightarrow\operatorname{Tor}(G,A_1\oplus A_2)$$

such that $\psi_1(<g,m,a_1>)=<g,m,(a_1,o)>$,

$\psi_2(<g,m,a_2>)=<g,m,(o,a_2)>$,

respectively. Hence there exists a group homomorphism

$$\psi\colon \operatorname{Tor}(G,A_1)\oplus\operatorname{Tor}(G,A_2)\longrightarrow\operatorname{Tor}(G,A_1\oplus A_2) \text{ such that}$$
$$\psi(<g_1,m_1,a_1>,\ <g_2,m_2,a_2>)=<g_1,m_1,(a_1,o)>+<g_2,m_2,(o,a_2)>.$$

Now it is easy to see that $\psi\circ\varphi=$identity map,

$\varphi\circ\psi=$identity map.

Therefore, φ is an isomorphism and

$$\operatorname{Tor}(G,A_1\oplus A_2)\cong\operatorname{Tor}(G,A_1)\oplus\operatorname{Tor}(G,A_2).$$ ///

Let R be a ring with 1. For a complex K of right (left) R-modules

$$K\colon \cdots\cdots\xrightarrow{\delta_{n-1}}K^n\xrightarrow{\delta_n}K^{n+1}\longrightarrow\cdots\cdots \qquad (n\in\mathbf{Z})$$

with $\delta_n\circ\delta_{n-1}=0$ and each K^n is a right (left) R-module, we put

$$\operatorname{Ker}\delta_n=C^n(K),\ \operatorname{Im}\delta_{n-1}=B^n(K).$$

Then we have

$$H^n(K)=C^n(K)/B^n(K).$$

Theorem 2.2. (Künneth Formula) In our situation if L is a complex of left R-modules and K a complex of right R-modules satisfying

$C^n(K)$ and $B^n(K)$ are flat modules for all n, $(※)_2$

then there is a short exact sequence of R-modules

$$0 \longrightarrow \sum_{p+q=n} H^p(K)\otimes_R H^q(L) \xrightarrow{\alpha} H^n(K\otimes_R L) \xrightarrow{\beta} \sum_{p+q=n+1} \mathrm{Tor}_1^R(H^p(K), H^q(L)) \longrightarrow 0 \quad (※)_3$$

for each dimension n.

Proof. Note that for

$$K:\ \cdots\cdots \longrightarrow K^{n-1} \xrightarrow{\delta^1_{n-1}} K^n \xrightarrow{\delta_n^1} K^{n+1} \longrightarrow \cdots\cdots$$

$$\cdot\ L:\ \cdots\cdots \longrightarrow L^{n-1} \xrightarrow{\delta^2_{n-1}} L^n \xrightarrow{\delta_n^2} L^{n+1} \longrightarrow \cdots\cdots$$

such that each K^n is a right R-module and each L^n a left R-module,

(i) $(K\otimes_R L)^n = \sum_{p+q=n} K^p \otimes_R L^q$

(ii) $\delta(k\otimes l) = \delta^1 k\otimes l + (-1)^{\deg k} k\otimes \delta^2 l$ $\qquad (k\otimes l \in K\otimes_R L)$

(iii) α: $H^p(K)\otimes_R H^q(L) \longrightarrow H^{p+q}(K\otimes_R L)$ is defined by
$\alpha(\mathrm{cls}(u) \otimes \mathrm{cls}(v)) = \mathrm{cls}(u\otimes v)$,

where $u\in C^p(K)$, $v\in C^q(L)$, the homology class of $u = \mathrm{cls}(u) \in H^p(K)$ and so on.

(iv) The exactness of $(※)_3$ is proved as follows;

If G is a flat right R-module and L a complex of left R-modules, then it is well-known that

$$\alpha': G\otimes H^n(L) \cong H^n(G\otimes L) \qquad (\otimes = \otimes_R). \qquad (※)_4$$

We put

$$C^n = C^n(K),\ D^n = K^n/C^n \cong B^{n+1}(K),$$

where K is a complex of right R-bimodules, then the complexes

$$C:\ \cdots\cdots \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow \cdots\cdots$$
$$D:\ \cdots\cdots \longrightarrow D^n \longrightarrow D^{n+1} \longrightarrow \cdots\cdots$$

are consisting of flat right R-modules with zero boundaries. (see$(※)_2$)

Consider the exact sequence of complexes

$$0 \longrightarrow C \longrightarrow K \longrightarrow D \longrightarrow 0 \qquad (0 \longrightarrow C^n \longrightarrow K^n \longrightarrow D^n \longrightarrow 0).$$

Since C is flat, the sequence of complexes

$$E:\ 0\longrightarrow C\otimes L\longrightarrow K\otimes L\longrightarrow D\otimes L\longrightarrow 0 \qquad (\otimes=\otimes_R)$$

is exact. The long exact sequence of cohomologies for E is

$$\cdots\longrightarrow H^{n-1}(D\otimes L)\xrightarrow{\varepsilon_{n-1}}H^n(C\otimes L)\longrightarrow H^n(K\otimes L)\longrightarrow H^n(D\otimes L)\xrightarrow{\varepsilon_n}H^{n+1}(C\otimes L)\longrightarrow\cdots,$$

where ε_{n-1} and ε_n are connecting homomorphisms. Therefore, for each $n\in\boldsymbol{Z}$ the sequence: $0\longrightarrow\mathrm{Coker}\ \varepsilon_{n-1}\longrightarrow H^n(K\otimes L)\longrightarrow\mathrm{Ker}\ \varepsilon_n\longrightarrow 0$ (※)5

is exact. Let δ'_{p-1}: $D^{p-1}\longrightarrow D^p$ be the homomorphism induced by δ'_{p-1}.

By the definition of $H^p(K)$, we have the exact sequence

$$S:\ 0\longrightarrow D^{p-1}\longrightarrow C^p\longrightarrow H^p(K)\longrightarrow 0$$

(Note that $D^{p-1}\cong B^p(K)\subset C^p$ and C^p is flat, and so is D^{p-1}).

Since D^{p-1} is flat, for $q\in\boldsymbol{Z}$ we have the exact sequence of R-bimodules.

$$S':\ 0\longrightarrow D^{p-1}\otimes H^q(L)\longrightarrow C^p\otimes H^q(L)\longrightarrow H^p(K)\otimes H^q(L)\longrightarrow 0 \quad (\otimes=\otimes_R).$$

From the long homology exact sequence for S', we get the exact sequence

$$\begin{array}{ccccccc} 0\longrightarrow \mathrm{Tor}_1^R(H^p(K),\ H^q(L)) & \xrightarrow{S_*} & D^{p-1}\otimes H^q(L) & \xrightarrow{\delta'\otimes 1} & C^p\otimes H^q(L) & \longrightarrow & H^p(K)\otimes H^q(L)\to 0 \\ & & \alpha'\downarrow\cong & & \alpha'\downarrow\cong & & \text{(see (※)4)} \\ & & H^{p+q-1}(D\otimes L) & \xrightarrow{\varepsilon_{p+q-1}} & H^{p+q}(C\otimes L), & & \end{array}$$

where S_* is the connecting homomorphisms of the exact sequence S'. (Note that $\mathrm{Tor}_1^R(C^p, H^q(L))=0$ since C^p is flat.)

Take the summation over $p+q=n$.

Then we have the exact sequence

$$\begin{array}{ccccccc} 0\to\Sigma\ \mathrm{Tor}_1^R(H^p(K),\ H^q(L)) & \xrightarrow{S_*} & \Sigma D^{p-1}\otimes H^q(L) & \xrightarrow{\delta'\otimes 1} & \Sigma C^p\otimes H^q(L) & \to & \Sigma H^p(K)\otimes H^q(L)\to 0 \\ & & \alpha'\downarrow\cong & & \cong\downarrow\alpha' & & \\ & & H^{n-1}(D\otimes L) & \xrightarrow{\varepsilon_{n-1}} & H^n(C\otimes L). & & \end{array}$$

But we can prove that the square in the above diagram is commutative ([13]). Therefore we have the following

$$\mathrm{Coker}\ \varepsilon_{n-1}\cong\mathrm{Coker}\ (\delta'\otimes 1)\cong\sum_{p+q=n}H^p(K)\otimes H^q(L)$$

$$\mathrm{Ker}\ \varepsilon_n \cong \mathrm{Ker}\ (\delta'\otimes 1) = \sum_{p+q=n+1} \mathrm{Tor}_1^R(H^p(K),\ H^q(L)).$$

Hence, by (※)$_5$, we have the exact sequence

$$0 \longrightarrow \sum_{p+q=n} H^p(K)\otimes H^q(L) \xrightarrow{\alpha} H^n(K\otimes L) \xrightarrow{\beta} \sum_{p+q=n+1} \mathrm{Tor}_1^R(H^p(K),\ H^q(L)) \longrightarrow 0,$$

where β is natural, since β can be described by the commutative diagram

$$\begin{array}{ccc} H^n(K\otimes L) & \xrightarrow{\ \beta\ } & \sum_{p+q=n+1} \mathrm{Tor}_1^R(H^p(K),\ H^q(L)) \\ \downarrow & & \downarrow S_* \\ H^n((K/C)\otimes L) & \xleftarrow[\alpha']{\ \cong\ } & \sum_{p+q=n} (K/C)^p \otimes H^q(L) \end{array} \qquad (※)_6$$

with the canonical projection $K \longrightarrow K/C$, α is an isomorphism, and is exact,

$$S_p:\ 0 \longrightarrow K^p/C^p \longrightarrow C^{p+1} \longrightarrow H^{p+1}(K) \longrightarrow 0,$$

where $K^p/C^p = D^p \cong B^{p+1}(K)$ and S_* is the sum of the corresponding connecting homomorphisms on Tor_1^R. (Note that each connecting homomorphism is natural.)

We have to note that if $C^n(L)$ and $B^n(L)$ are flat then symmetric arguments on L will produce a possibly different map β'. But, for complexes K and L of abelian groups we can prove that $\beta = \beta'$ ([3]). Hence we have proved the following.

Corollary 2.3. For comlexes K and L of abelian groups such that each K^n is torsion-free, the sequence

$$0 \longrightarrow \sum_{p+q=n} H^p(K)\otimes_R H^q(L) \xrightarrow{\alpha} H^n(K\otimes_R L) \xrightarrow{\beta} \sum_{p+q=n+1} \mathrm{Tor}_1^R(H^p(K),\ H^q(L)) \longrightarrow 0$$

is exact and splits by a homomorphism which is not natural.

By (※)$_1$, $\mathrm{Tor}_1^Z(H^p(K),\ H^q(L))$ is generated by

$$\{\langle \mathrm{cls}(u), m, cls(v)\rangle \mid m\in Z,\ cls(u)\in H^p(K),\ cls(v)\in H^q(L) \text{ and } \exists\ k\in K^{p-1},\ l\in L^{p-1} \text{ such that } \delta^1_{p-1}\, k = um,\ \delta^2_{p-1} l = mv\}.$$

In the Künneth Formula, for abelian groups the split exact sequence (※)$_3$ shows that the homology of $K\otimes L$ $(\otimes = \otimes_Z)$ is spanned by two types of cycles as follows;

A type *1* cycle is a cycle $u\otimes v$, where u is a cycle of K and v a cycle of L. That is, the classes of type *1* cycles is the image of α.

For a triple $\langle cls(u), m, cls(v)\rangle \in \mathrm{Tor}_1^Z(H^{p+1}(K),\ L^q(L))$ with $\delta^1 k = um$

and $\delta^2 l = mv$ for $k \in K^p$ and $l \in L^{p-1}$, the cycle

$$\frac{1}{m}\delta(k\otimes l) = u\otimes l + (-1)^p k\otimes v$$

is a type II cycle.
Note that $cls(u) \in H^{p+1}(K)$ and $cls(v) \in H^q(L)$ with $p+q=n$ imply that

(i) $u\otimes l \in K^{p+1}\otimes L^{q-1} \subset (K\otimes L)^n$,

(ii) $k\otimes v \in K^p \otimes L^q \subset (K\otimes L)^n$,

(iii) $k\otimes l \in K^p\otimes L^{q-1} \subset (K\otimes L)^{n-1}$.

Hence $\delta(k\otimes l) \in (K\otimes L)^n$ and since

$$\delta[(u\otimes l) + (-1)^p k\otimes v] = (-1)^{p+1} u\otimes v + (-1)^p u\otimes v = 0$$

it follows that

$\mathrm{cls}(\frac{1}{m}\delta(k\otimes l)) = \mathrm{cls}(u\otimes l + (-1)^p k\otimes v)) \in H^n(K\otimes L)$. It is easy to see that

$$\{\text{classes of type I cycles}\} \cap \{\text{classes of type II cycles}\} = \{0\}.$$

Therefore, we can define

$$\gamma:\ \mathrm{Tor}_1{}^Z(H(K),\ H(L)) \longrightarrow H(K\otimes L)/\alpha(H(K)\otimes H(L)) \qquad (H=H^*)$$

by $\gamma t = (-1)^p\ \mathrm{cls}(\frac{1}{m}\delta(k\otimes l))$, where $t = \langle \mathrm{cls}(u), m,\ \mathrm{cls}(v)\rangle$ such that there exist $k \in K^p$ and $l \in L^{p-1}$ satisfying $\delta^1 k = um$ and $\delta^2 l = mv$. That is, for $t \in \mathrm{Tor}_1{}^Z(H^{p+1}(K),\ H^q(L))$ $(H=H^*)$

$$\gamma t = \mathrm{cls}((-1)^p u\otimes l + k\otimes v)$$

(Note that $u\otimes l \in C^{p+1}(K)\otimes L^{q-1}$ and $k\otimes v \in K^p\otimes C^q(L)$).

Since $D = K/C$, the map

$$H(K\otimes L) \longrightarrow H(D\otimes L) \qquad (H=H^*)$$

carries $\gamma t = \mathrm{cls}((-1)^p\ u\otimes l + k\otimes v)$ into $\mathrm{cls}((k+C)\otimes v)$. ($(k+C)\otimes v$ is a cycle in $D\otimes L$ because $\delta^1 k \in C$). On the other hand, in $(※)_6$ we have

$$\alpha S_*(t) = \mathrm{cls}((k+\mathrm{C})\otimes v).$$

For each element $x = y + z \in H^n(K\otimes L)$, where $y = \gamma(t)$ and $z \in \alpha(\sum_{p+q=n} H^p(K)\otimes H^q(L))$, β is defined by $\beta(x) = t$.

Proposition 2.4. **Under the hypotheses of the Künneth Formula for abelian groups, γ is an isomorphism and β is the inverse of γ.**

3. Cohomology Spectral Sequences

Let A^* be a differential $\boldsymbol{Z}$-graded module (DG_Z-module) with a boundary operator

$$\delta: A^n \longrightarrow A^{n+1} \quad (n \in \boldsymbol{Z},\ \delta\delta = 0).$$

A filtration F^* of A^* is defined by a tower of differential $\boldsymbol{Z}$-graded submodule

$$\cdots \supset F^{p-1}A^* \supset F^p A^* \supset F^{p+1}A^* \supset \cdots \qquad (※※)_1$$

which is called a *descending filtration.* Note that

(1) $\forall n \in \boldsymbol{Z},\ F^p A^n \supset F^{p+1} A^n$
(2) $\delta(F^p A^{p+q}) \subset F^p A^{p+q+1}$.

Definition 3.1. A filtration F^* of a DG_Z-module A^* is said to be *bounded* if for each degree there exist integers $s = s(n) > t = t(n)$ such that

$$F^s A^n = 0, \quad F^t A^n = A^n.$$

That is, the filtraction of each A^n has limit length:

$$F^t A^n = A^n \supset F^{t+1} A^n \supset \cdots\cdots\cdots \supset F^s A^n = 0.$$

A filtration F^* of A^* is said to be *convergent above* if

$$\bigcup_p F^p A^* = A^*$$

and *bounded below* if for each n(degree) there exists an intege $s = s(n)$ such that $F^s A^n = 0$.

Definition 3.2. A $\boldsymbol{Z}$-bigraded module is a family

$$E = \{E^{p,q} \mid p, q = 0, \pm 1, \pm 2, \cdots\}$$

of $\boldsymbol{Z}$-modules. A differential $d: E \longrightarrow E$ of bidegree $(r, -r+1)$ is a family of homomorphisms

$$d: E^{p,q} \longrightarrow E^{p+r,\ q-r+1}$$

with $d^2 = 0$ for each p, q. The cohomology $H^*(E) = H^*(E, d)$ of E under this differential is the bigraded $\boldsymbol{Z}$-module $\{H^{p,q}(E)\}$ defined by

$$H^{p,q}(E) = \mathrm{Ker}(d: E^{p,q} \longrightarrow E^{p+r,\ q-r+1}) / dE^{p-r,\ q+r-1}.$$

A cohomology spectral sequence $E=\{E_r, d_r\}$ is a sequence $E_2, E_3, \cdots$ of $\mathbf{Z}$-bigraded modules, each with a differential

$$d_r: E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1} \quad (r=2,3,\cdots)$$

of bidegree $(r, -r+1)$ and with isomorphisms

$$H^*(E_r, d_r) \cong E_{r+1} \qquad (r=2,3,\cdots)$$

The bigraded module E_2 is called the *initial term* of this spectral sequence (Sometimes it is convenient to start the spectral sequence with $r=1$ and initial term E_1).

Consider the filtration (※※)$_1$ above. This filtration induces a filtration on the $\mathbf{Z}$-graded cohomology module $H^*(A^*)$, with $F^p(H^*(A^*))$ defined as the image of $H^*(F^p(A^*))$ under the injection $F^pA^* \longrightarrow A^*$.

That is, the filtration F^* of A^* determines a filtration F^pA^n of each A^n and the differential of A^* induces homomorphisms $\delta: F^pA^n \longrightarrow F^pA^{n+1}$ for each p and each n. The family $\{F^pA^n\}=\{F^pA^{p+q} \mid p+q=n\}$ is a $\mathbf{Z}$-bigraded module.

Definition 3.3. A cohomology spectral sequence $\{E_r, d_r\}$ is said to *converge* to a graded module H^*(in symbols, $E_2^p \Longrightarrow H^*$) if there exists a filtration F^* of H^* and for each p there exists an isomorphism $E_\infty^p \cong F^pH^*/F^{p+1}H^*$ of graded modules, where E_∞^p is defined as follows.

Let $\{E_r, d_r\}$ $(r=2,3,\cdots)$ be a cohomology spectral sequence. If $C^2=\text{Ker } d_2$ and $B^2=\text{Im} d_2$, then $E_3=C^2/B^2$. Since $E_4 \cong H^*(E_3, d_3)$, E_4 is isomorphic to a quotient group C^3/B^3 of C^2/B^2.

Therefore,

$$\text{Ker } d_3 = C^3/B^2, \ \text{Im} d_3 = B^3/B^2 \qquad (B^2 \subset B^3, \ C^3 \subset C^2)$$

and thus we get a sequence $0=B^1 \subset B^2 \subset \cdots \subset \cdots \subset C^2 \subset C^1 = E_2$ of bigraded subgroups of E_2 such that

① $E_{r+1}=C^r/B^r$

② $\text{Ker } d_r = C^r/B^{r-1}$, $\text{Im } d_r = B^r/B^{r-1}$.

We now put

$$C^\infty = \bigcap_{r=2}^{\infty} C^r, \qquad B^\infty = \bigcup_{r=2}^{\infty} B^r, \qquad (B^\infty \subset C^\infty),$$

and define $\quad E_\infty^p = C_\infty^p / B_\infty^p$

That is, $\quad E_\infty^{p,q} = C_\infty^{p,q}/B_\infty^{p,q} \qquad (E_\infty = \{E_\infty^{p,q}\})$.

Theorem 3.4. Each filtration F^* of a differential $\mathbf{Z}$-graded module A^* determines a cohomology spectral sequence $\{E_r, d_r\}$ $(r=1,2,\cdots)$ with natural isomorphisms

$$E_1^p \cong H^*(F^pA^*/F^{p+1}A^*);\ \text{i.e.,}\ E_1^{p,q} \cong H^{p+q}(F^pA^*/F^{p+1}A^*)$$

If F^* is bounded, then $E_2^p \Longrightarrow H^*(A^*)$.
That is,

$$E_\infty^p \cong F^p(H^*(A^*))/F^{p+1}(H^*(A^*));\ \text{i.e.,}$$
$$E_\infty^{p,q} \cong F^p(H^{p+q}(A^*))/F^{p+1}(H^{p+q}(A^*)).$$

Proof. We put

$$Z_r^p = \{a \in F^pA^* \mid \delta a \in F^{p+r}A^*\},$$

which is a submodule of F^pA^*. In particular, $Z_0^r = F^pA^*$, since $\delta F^pA^n \subset F^pA^{n+1} \subset F^pA^*$. Each Z_r^p is $\mathbf{Z}$-graded by degrees of A^*.
So we may regard Z_r as the bigraded $\mathbf{Z}$ module with

$$Z_r^{p,q} = \}a \in F^pA^{p+q} \mid \delta a \in F^{p+r}A^{p+q+1}\}.$$

Then our cohomology spectral sequence of the filtration F^* of A^* is defined by taking

$$E_r^p = (Z_r^p \cup F^{p+1}A^*)/(\delta Z_{r-1}^{p-r+1} \cup F^{p+1}A^*);$$

i.e.,

$$E_r^{p,q} = (Z_r^{p,q} \cup F^{p+1}A^{p+q})/(\delta Z_{r-1}^{p-r+1,\ q+r-2} \cup F^{p+1}A^{p+q}),$$

where $r=1,2,\cdots$ and while d_r: $E_r^p \longrightarrow E_r^{p+r}$ is the homomorphism induced on these subquotients by the differential δ: $A^* \longrightarrow A^*$.

Set $E_0^p = F^pA^*/F^{p+1}A^*$ and let η^p: $F^pA^* \longrightarrow E_0^p$ be the canonical projection. Before proceeding, we shall introduce the concept "additive relation" as follows.

Let R be a commutative ring with 1, and A and B be R-modules. An *additive relation* Υ: $A \longrightarrow B$ is a submodule of $A \oplus B$. The *inverse relation* γ^{-1}: $B \longrightarrow A$ is defined by

$$\gamma^{-1} = \{(b,a) \mid (a,b) \in \Upsilon \subset A \oplus B\} \subset B \oplus A.$$

For two additive relations γ: $A \longrightarrow B$ and ρ: $B \longrightarrow C$,
we also define the additive relation $\rho\Upsilon$: $A \longrightarrow C$ by $\delta\gamma = \{(a,c) \mid \exists\, b \in B$ such that $(a,b) \in \gamma,\ (b,c) \in \rho\}$

and we define the following:

$$\text{Def } \gamma=\{a\in A \mid \exists\, b\in B \text{ such that } (a,b)\in\gamma\},\ \text{Im}\gamma=\text{Def } \gamma^{-1},$$
$$\text{Ker } \gamma=\{a\in A \mid (a,o)\in\gamma\}, \qquad \text{Ind } \gamma=\text{Ker } \gamma^{-1}.$$

In particular, there is an isomorphiom ([13])

$$\text{Def } \gamma/\text{Ker } \gamma\cong\text{Im } \gamma/\text{Ind}\gamma \qquad (※※)_2$$

We now return to our proof. Consider the additive relation

$$E_o^{p-r}\xrightarrow{\delta^1}E_o^p\xrightarrow{\delta^2}E_o^{p+r}$$

which are induced on these subquotients by δ: $A^*\longrightarrow A^*$.
By our definitions, it follows that

$$\delta^2=\{(\eta^p a,\ \eta^{p+r}\delta a) \mid a\in Z_r^p\}$$
$$\delta^1=\{\eta^{p-r}a,\ \eta^p\delta a) \mid a\in Z_r^{p-r}\},$$

and thus we have the following:

$$\text{Def } \delta^2=\eta^p Z_r^p, \quad \text{Ker } \delta^2=\eta^p Z_{r+1}^p$$

$$\text{Im } \delta^1=\eta^p(\delta Z_r^{p-r}), \quad \text{Ind } \delta^1=\eta^p(\delta Z_{r-1}^{p-r+1})$$

(Note that ① $\forall a\in Z_{r+1}^p$, $\delta a\in F^{p+r+1}A^*\subset F^{p+r}A^*$ and $Z_{r+1}^p\subset Z_r^p$,

② $E_o^{p-r}=F^{p-r}A^*/F^{p-r+1}A^*$).

Since $\delta Z_{r-1}^{p-r+1}\subset\delta Z_r^{p-r}\subset Z_{r+1}^p\subset Z_r^p$ in view of inclusions, we define

$$E_r^p=(\eta^p Z_r^p)/\eta^p(\delta Z_{r-1}^{p-r+1}) \qquad (※※)_3$$

for each $r=0,1,2,\cdots$ (Note that if $r=0$ then $\eta^p Z_o^p/\eta^p(\delta Z_{-1}^{p+1})=F^pA^*/F^{p+1}A^*$). It is easy to see that δ induces homomorphisms

$$E_r^{p-r}\xrightarrow{d_r^1}E_r^p\xrightarrow{d_r^2}E_r^{p+r}$$

with

$$\text{Im } d_r^1=\eta^p(\delta Z_r^{p-r})/\eta^p(\delta Z_{r-1}^{p-r+1})$$

$$\text{Ker } d_r^2=\eta^p(Z_{r+1}^p)/\eta^p(\delta Z_{r-1}^{p-r+1}).$$

Since $\delta\delta=0$, it follows that Im $d_r^1 \subset$ Ker d_r^2 and

$$H^p(E_r,\ d_r) \cong \eta^p(Z_{r+1}^p)/\eta^p(\delta Z_r^{p-r}) = E_{r+1}^p.$$

Hence we have a cohomology spectral sequence. If $r=0$, then $Z_0^p=F^pA^*$ and d_0: $E_0^p \longrightarrow E_0^p$ is just the differential of the quotient complex $E_0^p=F^pA^*/F^{p+1}A^*$. This proves our first part of the theorem.

By (※※)$_2$, our cohomology spectral sequence defined as above can also be derived from the towers ([3] and [12])

$$\begin{array}{ccccccccccccc}
\delta Z_{-1}^{p+1} & \subset & \delta Z_0^p & \subset & \delta Z_1^{p-1} & \subset \cdots\cdots\cdots\cdots \subset & Z_2^p & \subset & Z_1^p & \subset & Z_0^p & = & F^pA^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \eta^p \\
B_0^p & \subset & B_1^p & \subset & B_2^p & \subset \cdots\cdots\cdots\cdots \subset & C_2^p & \subset & C_1^p & \subset & C_0^p & = & E_0^p,
\end{array}$$

where $C_r^p=\eta^p Z_r^p = Z_r^p/F^{p+1}A^*$ and $B_r^p=\eta^p\delta Z_{r-1}^{p-r+1}=\delta Z_{r-1}^{p-r+1}/F^{p+1}A^*$.

Consider the additive relation

$$\delta^2:\ F^pA^*/F^{p+1}A^* \longrightarrow F^{p+r}A^*/F^{p+r+1}A^*.$$

By (※※)$_2$, we have the isomorphism

$$\text{Def } \delta^2/\text{Ker } \delta^2 \cong \text{Im } \delta^2/\text{Ind } \delta^2.$$

Therefore, from the above tower, we have the isomorphism

$$C_r^p/C_{r+1}^p \cong B_{r+1}^{p+r}/B_r^{p+r}.$$

This gives d_r as the composite

$$E_r^p=(\eta^p Z_r^p)\ /\eta^p(\delta Z_{r-1}^{p-r+1}) = C_r^p/\ B_r^p \xrightarrow{\ \pi\ } C_r^p/C_{r+1}^p \cong$$
$$B_{r+1}^{p+r}/B_r^{p+r} \xrightarrow{\ \nu\ } C_r^{p+r}/B_r^{p+r} = E_r^{p+r},$$

where π is the canonical projection and ν is the injection.
This gives us the cohomology spectral sequence.

In order to prove our last part we put

$$C=\text{Ker } \delta(\text{cycles in } A^*),\ B=\text{Im } \delta(\text{boundaries in } A^*).$$

Then our filtration F^* induces filtrations $F^pC=C\cap F^pA^*$,
$F^pB=B\cap F^pA^*$ on C and B, respectively. By the description on the upper part of Definition 3.3, we have

$$H^*(F^pA^*)=F^p(H^*A^*)=(F^pC\cup B)/B.$$

Hence, by a *modular Noetherian isomorphism* ([13]), it follows that

$$F^p(H^*A^*)/F^{p+1}(H^*A^*)\cong(F^pC\cup B)/(F^{p+1}C\cup B)\cong F^pC/(F^{p+1}C\cup F^pB).$$

On the other hand,

$$F^pH^*/F^{p+1}H^*=F^p(H^*A^*)/F^{p+1}(H^*A^*)$$
$$\cong(F^pC\cup F^{p+1}A^*)/(F^pB\cup F^{p+1}A^*)\subset F^pA^*/F^{p+1}A^* \qquad (※※)_4$$

Since

$$E_r^p=(\eta^pZ_r^p)/\eta^p(\delta Z_{r-1}^{p-r+1}),$$

it follows from $(※※)_3$ that

the numerator of $E_r^p=(Z_r^p\cup F^{p+1}A^*)/F^{p+1}A^*\subset F^pA^*/F^{p+1}A^*$,

the denominator of $E_r^p=(\delta Z_{r-1}^{p-r+1}\cup F^{p+1}A^*)/F^{p+1}A^*$,

and thus

$$E_r^p=(Z_r^p\cup F^{p+1}A^*)/(\delta Z_{r-1}^{p-r+1}\cup F^{p+1}A^*)$$

i.e.,

$$E_r^{p,q}=(Z_r^{p,q}\cup F^{p+1}A^{p+q})/(\delta Z_{r-1}^{p-r+1,\ q+r-2}\cup F^{p+1}A^{p+q}).$$

Assume that the filtration F^* is bounded. Then by Definition 3.1, for each total degree $n=p+q$ there exist $t=t(n)$ and $s=s(n)$ such that $s>t$ and

$$F^tA^n=A^n\supset F^{t+1}A^n\supset\cdots\cdots\supset F^sA^n=0$$

Therefore, in the numerator of $E_r^{p,q}$, an element $a\in Z_r^{p,q}$ for r large has $\delta a\in F^{p+r}A^{p+r+1}=0$ $(p+r\geqq s)$, and hence $a\in F^pC^{p+q}$.

Therefore the numerators are $F^pC^{p+q}\cup F^{p+1}A^{p+q}$. As for the denominator, for r large every element in F^pB^{p+q} is the boundary of an element in $F^{p-r+1}A^*$, that is, of an element in Z_{r-1}^{p-r+1}.

Therefore the denominators equal $F^pB^{p+q}\cup F^{p+1}A^{p+q}$. Since E_∞ is defined as the intersection of numerators devided by union of denominators, we have

$$E_\infty^{p,q}=(F^pC^{p+q}\cup F^{p+1}A^{p+q})/(F^pB^{p+q}\cup F^{p+1}A^{p+q})\cong F^pH^*/F^{p+1}H^*$$

by $(※※)_4$. Therefore, it follows that $E_2^p\Longrightarrow H^*(A^*)$. ///

Example 3.5. Let $\mathscr{F}$ be a field, and let A be a differential $\mathbf{Z}$-graded $\mathscr{F}$-module (a vector space over $\mathscr{F}$). For a filtration F of A such that

$$\cdots\subset F_{p-1}A\subset F_pA\subset F_{p+1}A\subset\cdots,$$

there is a spectral sequence (E^r, d^r) such that

$$E_p{}^1\cong H(F_pA/F_{p-1}A),\ \text{i.e.,}\ E^1{}_{p,q}\cong H_{p+q}(F_pA/F_{p-1}A) \qquad ([9]).$$

Furthermore, if F is bounded then $E^2{}_p\Longrightarrow H(A)$ ([9]).
In this case, each $E^r{}_{p,q}$ is a vector space over $\mathscr{F}$. Thus for any vector space V over $\mathscr{F}$ we can consider the vector space $\mathrm{Hom}_{\mathscr{F}}(E^r{}_{p,q}, V)$, consider the semi-exact sequence

$$E^r{}_{p+r,q-r+1} \xrightarrow{d^r{}_{p+r,q-r+1}} E^r{}_{p,q} \xrightarrow{d^r{}_{p,q}} E^r{}_{p-r,q+r-1}$$

and the sequence

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathscr{F}}(E^r{}_{p-r,q+r-1}, V) & \xrightarrow{d_r{}^{p-r,q+r-1}} & \mathrm{Hom}_{\mathscr{F}}(E^r{}_{p,q}, V) & \xrightarrow{d_r{}^{p,q}} & \mathrm{Hom}_{\mathscr{F}}(E^r{}_{p+r,q-r+1}, V) \\ \Downarrow & & \Downarrow \quad \Downarrow & & \Downarrow \\ f & \longmapsto & f\circ d^r{}_{p,q},\quad g & \longmapsto & g\circ d^r{}_{p+r,q-r+1} \end{array}$$

(Note that $d_r{}^{p-r,q+r-1}(f)=f\circ d^r{}_{p,q}$). Then we see that

$$d_r{}^{p,q}\circ d_r{}^{p-r,q+r-1}=0.$$

In the above second sequence, we have isomorphisms

$$\begin{aligned} \mathrm{Ker}\ d_r{}^{p,q}/\mathrm{Im}\ d_r{}^{p-r,q+r-1} &\cong \mathrm{Hom}_{\mathscr{F}}(\mathrm{Ker}\ d^r{}_{p,q}/\mathrm{Im}\ d^r{}_{p+r,q-r+1}, V) \\ &\cong \mathrm{Hom}_{\mathscr{F}}(E^{r+1}_{p,q}, V) \end{aligned}$$

because that

$$\begin{aligned} &\mathrm{Ker}\ d_r{}^{p,q}=\{f: E^r{}_{p,q}\longrightarrow V \mid f \text{ is linear and } f|\mathrm{Im}\ d^r{}_{p+r,q-r+1}=0\ \} \\ &\mathrm{Im}\ d_r{}^{p-r,q+r-1}=\{g: E^r{}_{p,q}\longrightarrow V \mid g \text{ is linear and } g|\mathrm{Ker}\ d^r{}_{p,q}=0\ \} \end{aligned}$$

In detail, we have the isomorphism

$$\begin{array}{ccc} \mathrm{Ker}\ d_r{}^{p,q}/\mathrm{Im}\ d_r{}^{p-r,q+r-1} & \xrightarrow{\ \cong\ } & \mathrm{Hom}_{\mathscr{F}}(\mathrm{Ker}\ d^r{}_{p,q}/\mathrm{Im}\ d^r{}_{p+r,q-r+1}, V) \\ \Downarrow & & \Downarrow \\ [f] & \longmapsto & f|\mathrm{Ker}\ d^r{}_{p,q} \end{array}$$

Note that ① for any subspace U of a vector space W over $\mathscr{F}$ and for each $f\in\mathrm{Hom}_{\mathscr{F}}(U,$

V), there exists an extension $\tilde{f}\in \mathrm{Hom}_{\mathscr{F}}(W,V)$ of f and ② for each

$$g\in \mathrm{Hom}_{\mathscr{F}}\,(\mathrm{Ker}\ d^{r}{}_{p,q}/\mathrm{Im}\ d^{r}{}_{p+r,q-r+1},V)$$

and for any two extensions $\tilde{g}_1$ and $\tilde{g}_2\in \mathrm{Ker}\ d_r^{p,q}/\mathrm{Im}\ d_r^{p-r,q+r-1}$ of g,

we have $\quad \tilde{g}_1-\tilde{g}_2\in \mathrm{Im}\ d_r^{p-r,q+r-1}$

Let us put

$$E_r^{p,q}\cong \mathrm{Hom}_{\mathscr{F}}(E_r^{p,q},V).$$

Then, by the above reason,

$$E^{p,q}_{r+1}\cong \mathrm{Ker}\ d_r^{p,q}/\mathrm{Im}\ d_r^{p-r,q+r-1}\cong \mathrm{Hom}_{\mathscr{F}}(E^{r+1}_{p,q},V).$$

Therefore, we get a cohomology spectral sequence $\{E_r, d_r\}$ from $\{E^r, d^r\}$. In general, when a spectral sequence $\{E^r, d^r\}$ of vector spaces over a field $\mathscr{F}$ is given, we can prove that $\{E_r=\mathrm{Hom}_{\mathscr{F}}(E^r, V), d_r\}$ (V is a vector space over $\mathscr{F}$) is a cohomology spectral sequence by the same method above. ///

Proposition 3.6. If a filtration F^* of $\boldsymbol{Z}$-graded module A^* is bounded below and convergent above, then $E_2^p\Longrightarrow H^*(A^*)$.

Proof. As in the prove of Theorem 3.4, we put

$$C=\mathrm{Ker}\ \delta, \qquad B=\mathrm{Im}\ \delta.$$

Then the intersection of the numerators of E_r^p is $F^pC\bigcup F^{p+1}A^*$ since F^* is bounded below. Each element of F^pB is a boundary δa for some $a\in A^*=\bigcup_t F^tA^*$, hence $a\in F^tA^*$ for some t, since F^* is convergent above. Thus $a\in Z^{p-r+1}_{r-1}$ for $r=t+p-1$, so $F^pB\bigcup F^{p+1}A^*$ is again the union of the denominators $\delta Z^{p-r+1}_{r-1}\bigcup F^{p+1}A^*$, and thus we have

$$E_2^p\Longrightarrow H^*(A^*)$$

(for details see the last part of the proof of Theorem 3.4)
Therefore, even if F^* is not bounded we have the following:

F^* is bounded below and convergent above $\Longrightarrow$
F^* gives the convergence $E_2^p\Longrightarrow H^*(A^*)$. ///

4. Cohomology Spectral Sequences of Bicomplexes.

Let a bicomplex K be a family $\{K^{p,q}\}$ of modules with two families

$$\delta': K^{p,q} \longrightarrow K^{p+1,q}, \quad \delta'': K^{p,q} \longrightarrow K^{p,q+1}$$

of module homomorphisms, defined for all integers p and q and such that

$$\delta'\delta' = 0 = \delta''\delta'', \quad \delta'\delta'' + \delta''\delta' = 0. \qquad (※※※)_1$$

Thus K is a $\mathbf{Z}$-bigraded module and δ', δ'' are module homomorphisms of bidegrees (1.0) and (0.1), respectively.

Definition 4.1. A bicomplex K is *positive* if $K^{p,q}=0$ unless $p \geq 0$ and $q \geq 0$. (Note that each object $K^{p,q}$ in a bicomplex K may be R-modules where R is a commutative ring with 1), *Λ-modules* where Λ is a commutative algebra with 1, graded modules or objects from some abelian category. We define

$$H_2^{p,q}(K) = \mathrm{Ker}(\delta'': K^{p,q} \longrightarrow K^{p,q+1}) / \mathrm{Im}\ (\delta'': K^{p,q-1} \longrightarrow K^{p,q}),$$

Then it is a bigraded object with a differential

$$\delta': H_2^{p,q}(K) \longrightarrow H_2^{p+1,q}(K)$$

which is induced by the original δ'. We also define

$$H_1^p H_2^q(K) = \mathrm{Ker}(\delta': H_2^{p,q}(K) \longrightarrow H_2^{p+1,q}(K) / \mathrm{Im}(\delta': H_2^{p-1,q}(K) \longrightarrow H_2^{p,q}(K)),$$

which is a bigraded object. Similarly, the *iterated* homology $H_2^q H_1^p(K)$ are defined. Each bicomplex K is defined as a single complex $X = \mathrm{Tot}(K)$ such that

$$X^n = \sum_{p+q=n} K^{p,q}, \quad \delta = \delta' + \delta'': X^n \longrightarrow X^{n+1} \qquad (※※※)_2$$

Then it follows from $(※※※)_1$ that $\delta\delta = 0$. If K is positive, so is X, and in this case each direct sum in $(※※※)_2$ is finite.

For example, let R be a commutative ring with 1, and let X and Y be complexes of R-modules such that

$$X: \cdots\cdots \longrightarrow X^n \xrightarrow{\delta'} X^{n+1} \longrightarrow \cdots\cdots,$$
$$Y: \cdots\cdots \longrightarrow Y^n \xrightarrow{\delta''} Y^{n+1} \longrightarrow \cdots\cdots.$$

Then $X \otimes Y (\otimes = \otimes_R)$ is a bicomplex $\{X^p \otimes Y^q\}$ with boundary operators

(i) By Definition 4.2, an element $a \in (F^p X)^n$ has the form
$a = a^{p,q} + a^{p+1,q-1} + a^{p+2,q-2} + \cdots, a^{p,q} \in K^{p,q}$ and $p+q=n$.

Thus we have

$$\delta a = \delta'' a^{p,q} + (\delta' a^{p,q} + \delta'' a^{p+1,q-1}) + (\delta' a^{p+1,q-1} + \delta'' a^{p+2,q-2}) + \cdots,$$

where we have grouped terms of the same bidegree. Therefore,

(i) $\delta'' a^{p,q} = 0 \Longleftrightarrow a \in Z_1^{p,q}$,

(ii) $\delta'' a^{p,q} = 0 = \delta' a^{p,q} + \delta'' a^{p+1,q-1} \Longleftrightarrow a \in Z_2^{p,q}$.

Since

$$E_2'^{p,q} = (\eta^p Z_2^{p,q})/(\eta^p \delta Z_1^{p-1,q}) \text{ for each } a^{p,q} \in L_2^{p,q},$$

we have

$$a^{p,q} \equiv a^{p,q} + a^{p+1,q-1} \pmod{F_1^{p+1} X}$$

and

$$\delta(a^{p,q} + a^{p+1,q-1}) = \delta' a^{p+1,q-1} \in F_1^{p+2} X.$$

Hence $a^{p,q} \in L_2^{p,q} \Longleftrightarrow [a^{p,q}] = a^{p,q} \in \eta^p Z_2^{p,q}$, and so we have $L_2^{p,q} = \eta^p Z_2^p$. Next, suppose that an element

$$b = b^{p-1,q} + b^{p,q-1} + b^{p+1,q-2} + \cdots\cdots \quad \text{is contained in } (F_1^{p-1} X)^{n-1}$$

If $\delta'' b^{p-1,q} = 0$ and $\delta' b^{p-1,q} + \delta'' b^{p,q-1} = 0$, then

$$\delta b = (\delta' b^{p,q-1} + \delta'' b^{p+1,q-2}) + \cdots\cdots \quad \text{is contained in } (F_1^p X)^n.$$

Hence we see that $b \in Z_1^{p-1,q}$. In this case,

$$\eta^p \delta b = \delta' b^{p-1,q} + \delta'' b^{p,q-1},$$

and thus $M_2^{p,q} = \eta^p \delta Z_1^{p-1,q} \subset \eta^p Z_2^{p,q}$ ([13]). In consequence, we have

$$E_2^{p,q} = L_2^{p,q} / M_2^{p,q}.$$

(ii) For each $a^{p,q} \in L_3^{p,q}$ if we put
$a = a^{p,q} + a^{p+1,q-1} + a^{p+2,q-2} \in (F_1^p X)^n$.

$$\delta'(x\otimes y)=\delta' x\otimes y,\quad \delta''(x\otimes y)=(-1)^{\deg x}\ x\otimes\delta'' y$$

$(x\otimes y\in X\otimes Y)$. It is easy to prove that δ' and δ'' defined as above satisfy (※※※)$_1$. In this case,

$$\mathrm{Tot}(X\otimes Y)\ni(X\otimes Y)^n=\sum_{p+q=n} X^p\otimes Y^q,\quad \delta=\delta'+\delta''.$$

Definition 4.2. The *first filtration* $F_I{}^*$ of $X=\mathrm{Tot}(K)$ is defined by the subcomplexes $F_I{}^p$ such that

$$(F_I{}^pX)^n=\sum_{k\geq p}K^{k,n-k}. \qquad (※※※)_3$$

Then we have the following:

(i) $(F_I{}^pX)^n\subseteq(F_I{}^pX)^{n+1}$,

(ii) $\cdots\supset(F_I{}^pX)^n\supset(F_I{}^{p+1}X)^n\supset\cdots\supset\cdots$

(iii) $X^n=\bigcup_p(F_I{}^pX)^n$.

In this case, by theorem 3.4, we have the cohomology spectral sequence $\{E_r', d_r\}$ which is called the *first cohomology spectral sequence* of the filtration F_I.

Theorem 4.3. For the first cohomology spectral sequence $\{E_r', d_r\}$ of a bicomplex K with associated total complex X, we have the following $(p+q=n)$:

(i) $E_2'^{p,q}=L_2{}^{p,q}/M_2{}^{p,q}$, where

$L_2{}^{p,q}=\{a^{p,q}\in K^{p,q}\,|\,\delta''a^{p,q}=0$ and $\exists\ a^{p+1,q-1}\in K^{p+1,q-1}$ such that $\delta'a^{p,q}$ $=\delta''a^{p+1,q-1}\}$

$M_2{}^{p,q}=\{\delta'b^{p-1,q}+\delta''b^{p,q-1}\in K^{p,q}\,|\,\delta''b^{p-1,q}=0,\ b^{p-1,q}\in K^{p-1,q},\ b^{p,q-1}\in K^{p,q-1}\}$

(ii) $E_3'^{p,q}=L_3{}^{p,q}/M_3{}^{p,q}$ where

$L_3{}^{p,q}=\{a^{p,q}\in K^{p,q}\,|\,\delta''a^{p,q}=0,\ \exists\ a^{p+1,q-1}\in K^{p+1,q-1}$ such that $\delta'a^{p,q}=-\delta''a^{p+1,q-1}$ and $\exists a^{p+2,q-2}\in K^{p+2,q-2}$ such that $\delta'a^{p+1,q-1}=-\delta''a^{p+2,q-2}\}$

$M_3{}^{p,q}=\{\delta'b^{p-1,q}+\delta''b^{p,q-1}\,|\,\exists b^{p-2,q+1}\in K^{p-2,q+1}$ such that $\delta''b^{p-2,q+1}=0$ and $\exists\ b^{p-1,q}\in K^{p-1,q}$ such that $\delta'b^{p-2,q+1}=-\delta''b^{p-1,q},\ b^{p,q-1}\in K^{p,q-1}\}$

(iii) $E_2'^{p,q}\cong H_I{}^pH_{II}{}^q(K)$.

Proof. Recall that $E_r{}^p=(\eta^pZ_r{}^p)/\eta^p\delta Z_{r-1}^{p-r+1}$ (see(※※)$_3$) in the proof of theorem 3.4.

Then we have $\delta a=\delta' a^{p+2,q-2}\in(F_1^{p+2}X)^{n+1}$.
It follows that $a\in Z_3^{p,q}$ and $a\equiv a^{p,q}$ mod $(F_1^{p+1}X)^n$
implies that $a^{p,q}\in L_3^{p,q}\Longleftrightarrow[a^{p,q}]=a^{p,q}\in\eta^p Z_3^{p,q}$. Similarly, suppose that an element

$$b=b^{p-2,q+1}+b^{p-1,q}+b^{p,q-1}+b^{p+1,q-2}+\cdots \text{ is contained in } (F_1^{p-1}X)^{n-1}$$

If $\delta'' b^{p-2,q+1}=0$ and $\delta' b^{p-2,q+1}=-\delta'' b^{p-1,q}$, then

$$\delta b=(\delta' b^{p-1,q}+\delta'' b^{p,q-1})+\cdots \text{ is contained in } (F_1^pX)^n,$$

and thus $b\in Z_2^{p-2,q+1}$. Since

$$\eta^p\delta b=\delta' b^{p-1,q}+\delta'' b^{p,q-1}$$

we have $M_3^{p,q}=\eta^p\delta Z_2^{p-2,q+1}\subset\eta^p Z_3^{p,q}$ ([13]). Hence it follows that

$$E_3^{p,q}=L_3^{p,q}/M_3^{p,q}.$$

(iii) Recall the proof (i) of this theorem. In L_2 the first condition on $a^{p,q}$ makes it a δ''-cycle, and thus it determines (cls$''$ $a^{p,q}$) $\in H_2^{p,q}(K)$; the second condition asserts that this homology class (cls$''$ $a^{p,q}$) lies in the kernel of δ': $H_2^{p,q}(K)\longrightarrow H_2^{p+1,q}(K)$. The term $\delta'' b^{p-1,q-1}$ in M_2 can vary $a^{p,q}$ by a δ''-boundary, leaving (cls$''$ $a^{p,q}$) unchanged; the term $\delta' b^{p-1,q}$ can vary (cls$''$ $a^{p,q}$) by δ'(cls$''$ $b^{p-1,q}$). Therefore the correspondence

$$\begin{array}{ccc} L_2^{p,q}/M_2^{p,q} & \longrightarrow & H_1^pH_2^q(K) \\ \Cup & & \Cup \\ [a^{p,q}] & \longmapsto & \mathrm{cls}'(\mathrm{cls}''\, a^{p,q}) \end{array}$$

provides the desired isomorphism $E_2^{p,q}\cong H_1^pH_2^q(K)$. ///

Corollary 4.4. Under the situation of Theorem 4.3, if $K^{p,q}=0$ for $p>0$ then $E_2'\Longrightarrow H(X)$. If K is positive, the E' lies in the first quadrant.

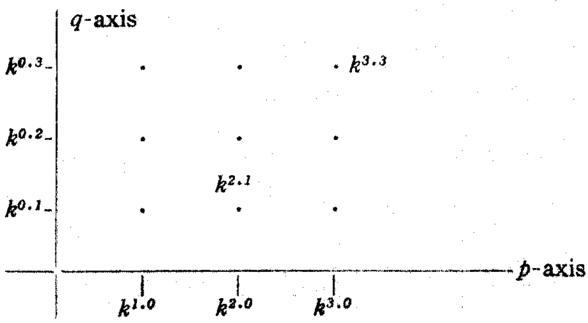
Proof. By (※※※)$_3$, we have

$$X^n=\bigcup_p(F_1^pX)^n,$$

and thus the filtration F_1 is convergent above. The fact that $K^{p,q}=0$ for $p>0$ means $F_1'X=0$. Hence the filtration F_1 is bounded below.

By Proposition 3.6, $E_2'^p\Longrightarrow H(X)$.

Next we assume that K is positive. Then $K^{p,q}$ lies in the first quadrant as follows.



Therefore, by

$$E_2'^{p,q} \cong H_I^p H_{II}^q(K) \quad \text{((iii) of theorem 4.3)}$$

E' lies in the first quadrant. ///

Theorem 4.5. Let X and Y be complexes of abelian group with each X^n a free group such that

$$X: \cdots \longrightarrow X^n \xrightarrow{\delta'} X^{n+1} \longrightarrow \cdots,$$

$$Y: \cdots \longrightarrow Y^n \xrightarrow{\delta''} Y^{n+1} \longrightarrow \cdots.$$

In the first cohomology spectral sequence (see Definition 4.2) of the bicomplex $K = X \otimes Y$ ($\otimes = \otimes_Z$), we have

$$E_2^{p,q} = E_\infty^{p,q} \cong H_I^p(X \otimes H_{II}^q(Y)).$$

Proof. As before, we put

$$\delta'(x \otimes y) = \delta' x \otimes y, \quad \delta''(x \otimes y) = (-1)^{\deg x} x \otimes \delta'' y$$
$$\delta = \delta' + \delta'' \qquad (x \otimes y \in X \otimes Y).$$

Then, by the first filtration F_I^*, we get

$$E_2^{p,q} \cong H_I^p H_{II}^q(K) \quad \text{(see (iii) of Theorem 4.3 and } E \text{ is } E' \text{ in Theorem 4.3).}$$

In the exact sequence of abelian groups,

$$0 \longrightarrow \text{Im } \delta'' \longrightarrow \text{Ker } \delta'' \longrightarrow H_{II}^q(Y) \longrightarrow 0.$$

The sequence

$$0 \longrightarrow X^p \otimes \text{Im } \delta'' \longrightarrow X^p \otimes \text{Ker } \delta'' \longrightarrow X^p \otimes H_{II}^q(Y) \longrightarrow 0$$

is exact since X^n is free. This implies that

$$H_2^{p,q}(K)=X^p\otimes H_2^q(Y).$$

Hence, $H_1^pH_2^q(K)=H_1^p(X\otimes H_2^q(Y))$ and thus

$$E_2^{p,q}\cong H_1^p(X\otimes H_2^q(Y)).$$

Next, we recall the Künneth for abelian groups in Theorem 2.2.

$$H_1^p(X)\otimes H_2^q(Y)\rightarrowtail\xrightarrow{\alpha} H_1^p(X\otimes H_2^q(Y)) \xrightarrow{\beta}\!\!\!\to \mathrm{Tor}_1(H_1^{p+1}(X), H_2^q(Y))$$

which is split since each X^n is free, where

$H_2^q(Y)\colon 0\longrightarrow H_2^q(Y)\longrightarrow 0$ is a complex.

Therefore, as in Proposition 2.2, each element of $H_1^p(X\otimes H_2^q(Y))$ can be described by

$$(\mathrm{cls}(u)\otimes \mathrm{cls}(v))+\mathrm{cls}\ ((-1)^pu'\otimes l+k\otimes v'),$$

where $\mathrm{cls}(u)\in H_1^p(X)$, $\mathrm{cls}(v)\in H_2^q(Y)$, $k\in X^p$, $u'\in X^{p+1}$, $l\in Y^{q-1}$, $v'\in Y^q$ and there exists an integer m such that

$$\delta'k=u'm \text{ and } \delta''l=mv'.$$

Then

$$u\otimes v+k\otimes v'\in X^p\otimes Y^q\subset(F^pK)^n,\ u'\otimes l\in X^{p+1}\otimes Y^q\subset(F^{p+1}K)^n,$$

and thus

$$\begin{aligned}\delta((u\otimes v+k\otimes v')+((-1)^pu'\otimes l))&=\delta'k\otimes v'+(-1)^{p+(p+1)}u'\otimes\delta''l\\ &=u'm\otimes v'-u'\otimes mv'=0.\end{aligned}$$

Since the homomorphism

$$\begin{array}{ccc} d_2^{p,q}\colon E_2^{p,q} & \longrightarrow & E_2^{p+2,q-1} \\ \wr\| & & \wr\| \\ H_1^p(X\otimes H_2^q(Y)) & \longrightarrow & H_1^{p+2}(X\otimes H_2^{q-1}(Y)) \end{array}$$

is induced from δ, by (※※)$_3$ in §3 we get $d_2^{p,q}=0$. This implies that $d_2=d_3=\cdots=0$ and $E_2=E_\infty$. ///

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