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The Moment Problem and C^n-Scalar Operators

by

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> ***Abstract***<

We show that a bounded linear operator, T, on a Banach space, X, is C^n-scalar if the sepuence $\{\frac{k!}{(k+n)!}\phi(T^{k+n}x)\}_{k=0}^{\infty}$ is positive-definite, for sufficiently many ϕ in X^*, x in X. We use this to show that $(T_n f)(t)\equiv tf(t)+nJf(t)$, where $Jf(t)=\int_0^t f(s)ds$, is C^n-scalar on $L^p([0,1],v)$, for $1\leq p\leq\infty$, for a large class of measures, v. Other corollaries include the spectral theorem for bounded symmetric operators on a Hilbert space.

Introduction

An operator is C^n-*scalar* if it has a functional calculus defined for any function with n continuous derivatives. Even in finite dimensions, this provides an interesting generalization of self-adjoint operators. The 2×2 matrix

$A\equiv\begin{bmatrix} a & 1\\ 0 & a\end{bmatrix}$ is C^1-scalar, with $f(A)\equiv\begin{bmatrix} f(a) & f'(a)\\ 0 & f(a)\end{bmatrix}$, defining an algebra homomorphism. A similar construction, along with the Jordan canonical form for $n\times n$ matrices, shows that any $n\times n$ matrix is C^{n-1}-scalar. (See [3]).

The *moment problem*, from classical analysis, asks the following question: given A, a subset of the complex plane, for which sequences $\{a_n\}_{n=0}^{\infty}$ will there exist a positive measure μ such that $a_n=\int_A t^n d\mu(t)$, for all n? Simple answers have been given for $A=R$, $[0,\infty]$, $[0,1]$, or the unit circle. (See [8]).

Connections between moment theory and the spectral theorem have been observed many times. For example, in [1] and [7], the spectral theorem is proved using moment theory; in [6], some solutions to the moment problem are proven using the spectral theorem.

In thls paper, we use the moment problem to get sufficient counditions for an operator, T, to be C^n-scalar on $[-\|T\|,\|T\|]$. (Proposition 7) The case $n=0$, on a Hilbert space, gives the spectral theorem (Corollary 11). A sufficient condition for T to be C^n-scalar is that the sequence $\{\frac{k!}{(k+n)!}\phi(T^{k+n}x)\}_{k=0}^{\infty}$ be positive-definite, for "sufficiently many" ϕ in X^*, x in X. The precise meaning of "sufficiently many" is contained in the definition of a "determining set" (definition 3).

When X is a Banach lattice, it is sufficient to have the sequence above be positive-definite whenever ϕ and x are positive. (Corollary 9). When X is a Hilbert space, with inner product "$<\ >$", it is sufficient to have the real and imaginary parts of the sequence $\{\frac{k!}{(k+n)!}<x,\ T^{k+n}x>\}_{k=0}^{\infty}$ be positive-definite, for all x in H (Corollary 10). Thus, the proof of the spectral theorem (Corollary 11) merely involves showing that $\{<x,\ T^k x>\}_{k=0}^{\infty}$ is positive-definite whenever T is symmetric.

We apply our results to an operator first considered by Kantorovitz ([4],[5]), $(T_n f)(t)\equiv tf(t)+n\int_0^t f(s)\ ds$. He showed that T_n is C^n-scalar on $L^p[0,1]$, for $1\leq p<\infty$. We will extend this to $p=\infty$, and to a large class of measures. We will show that T_n is C^n-scalar on $L^p([0,1]),v)$, for $1\leq p\leq\infty$, whenever Lebesgue measure is absolutely continuous with respect to v, with a bounded Radon-Nikodym derivative, and v is finite.

All operators are bounded and linear, on a Banach space, usually labelled "X".

Definition 1. If D is a closed, bounded subset of the real line, then $C^n(D)$ is defined to be the set of all f: $D\to C$ with n continuous derivatives. We will use the norm

$$(1) \qquad \|f\|_{n,\infty}\equiv\sup_{k\leq n}\|f^{(k)}\|_\infty\equiv\sup_{x\in D,\,k\leq n}\{|f^{(k)}(x)|\}$$

where $f^{(k)}$ is the k^{th}-derivitive of f.

Definition 2. T is *C^n-scalar* on D if there exists Λ: $C^n(D)\to B(X)$, a continuous algebra homomorphism, such that $\Lambda f_0=I$, $\Lambda f_1=T$, where $f_0(t)\equiv 1$, $f_1(t)\equiv t$. Λf is often written "$f(T)$".

T is C^n-scalar on [a,b], a bounded interval, if and only if there exists $M<\infty$ such thar $\|p(T)\|\leq M\|p\|_{n,\infty}$, for all polynomials p, because the polynomials are dense in C^n ([a,b]).

Definition 3. $\alpha\subseteq X\times X^*$ is a *determining set* it there exists $M<\infty$ such that whenever T is in $B(X,X^{**})$, with $|(Tx)(\phi)|\leq\|x\|\|\phi\|$, for all (x,ϕ) in α then $\|T\|\leq M$.

Theorem 4. Suppose there exists $M<\infty$ such that for all(x,ϕ) in α, a determining set, there exists $\mu_{\phi,x}$, a complex-valued measure, such that $\|\mu_{\phi,x}\|$, the total variation of $\mu_{\phi,x}$, is less tann $M|\mu_{\phi,x}$ [a, b]|, and for all k

$$\frac{k!}{(k+n)!}\phi(T^{k+n}x)=\int_a^b t^k d\mu_{\phi,x}(t).$$

Then T is C^n-scalar on [a, b]·

Proof. If p is a polynomial, then

$$\phi(p(T)x)=\sum_{i=0}^{n-1}\frac{p^{(i)}0}{i!}\phi(T^i x)+\int_a^b p^{(n)}(t)du_{\phi,x}(t),$$

for all (x,ϕ) in α, so that

$$|\phi(p(T)x)|\leq\sum_{i=0}^{n-1}\frac{|p^{(i)}(0)|}{i!}|\phi(T^i x)|+\|p^{(n)}\|_\infty\|\mu_{\phi,x}\|$$

$$\leq\sum_{i=0}^{n-1}\frac{|p^{(i)}(0)|}{i!}\|\phi\|\|T^i\|\ \|x\|+\|p^{(n)}\|_\infty(M|\mu_{\phi,x}[a,b]|)$$

$$\leq\|p\|_{n,\infty}(\sum_{i=0}^{n-1}\frac{\|\phi\|\ \|T^i\|\ \|x\|}{i!}+\frac{M}{n!}|\phi(T^n x)|)$$

$$\leq\|p\|_{n,\infty}\|\phi\|\ \|x\|(\sum_{i=0}^{n-1}\frac{\|T^i\|}{i!}+\frac{M\|T^n\|}{n!}).$$

Since α is a determining set, there exists K such that $\|p(T)\|\leq K\|p\|_{n,\infty}$. Since p was an arbitrary polynomial, T is C^n-scalar on [a, b].

Definition 5. A sequence $\{a_n\}_{n=0}^\infty$ is *positive definite* if $\sum\alpha_k\alpha_j a_{k+j}\geq 0$, for all finite sequences $\{a_k\}$ of complex numbers.

There exists a positive measure μ such that $a_n=\int_R t^n d\mu(t)$, for all n, if and only if $\{a_n\}$ is positive-definite. (See [8]).

We will also need the following elementary lemma.

Lemma 6. Suppose $a_n=\int_R t^n du(t)$, where μ is positive and there exists M, r so that $|a_n|\leq Mr^n$, for all n. Then μ is unique and supported on $[-r,r]$.

Proposition 7. Suppose the real and imaginary parts of the sequence $\{\frac{k!}{(k+n)!}\phi(T^{k+n}x)\}_{k=0}^\infty$ are positive-definite, for all (x,ϕ) in α, a determining set. Then T is C^n-scalar on $[-\|T\|, \|T\|]$.

Proof. For all (x,ϕ) in α, since $\left|\frac{k!}{(k+n)!}\phi(T^{k+n}x)\right| \le \|\phi\|\,\|x\|\,\|T^n\|(\|T\|)^k$, for all k, there exists unique positve measures $m_{\phi,x}$ and $v_{\phi,x}$ such that

$$\frac{k!}{(k+n)!}\phi(T^{k+n}x) = \int_{-\|T\|}^{\|T\|} t^k d(m_{\phi,x}+iv_{\phi,x})(t).$$

Let $\mu_{\phi,x} \equiv m_{\phi,x}+iv_{\phi,x}$. Since $\|\mu_{\phi,x}\| \le 2|\mu_{\phi,x}[-\|T\|,\|T\|]|$, the result follows from Theorem 4.

Corollary 8. Suppose X is reflexive, and the real and imaginary parts of $\{\phi(T^k x\}_{k=0}^{\infty}$ are positive-definite, for all (x,ϕ) in α, a determining set. Then T is a spectral operator of scalar type.

Proof. When X is reflexive, T is a spectral operator of scalar type if and only if T is C^0-scalar. (See [2]).

Two examples of determining sets are given in the following two corollaries of Proposition 7.

A complex Banach lattice, X, may be formed by taking a real Banach lattice, Y, and letting $X \equiv \{x+iy \mid x,y$ are in $Y\}$, with the norm on X having the property that $\|x+iy\| \ge \max(\|x\|, \|y\|)$, for all x,y in Y, and X^{+}' the positive cone of X, be Y^+.

Corollary 9. Suppose X is a Banach lattice, and the real and imaginary parts of $\{\frac{k!}{(k+n)!}\phi(T^{k+n}x)\}_{k=0}^{\infty}$ are positive-definite, for all positive ϕ, x. Then T is C^n-scalar on $[-\|T\|, \|T\|]$.

Proof. We need to show that $(X^*)^+ \times X^+$ is a determining set(See definition 3). So suppose T is in $B(X, X^{**})$, with $|(Tx)(\phi)| \le \|x\|\,\|\phi\|$, for all positive x in X, postive ϕ in X^*. If ψ is in X^*, then there exist ψ_1, ψ_2, ψ_3, ψ_4, in $(X^*)^+$, such that ψ_1 is orthogonal to ψ_2, ψ_3 is orthogonal to ψ_4, and

$$\psi = (\psi_1-\psi_2)+i(\psi_3-\psi_4).$$

$|(Tx)\psi| \le \sum_{i=1}^{4} |Tx(\psi_i)| \le \sum_{i=1}^{4} \|x\|\,\|\psi_i\| \le 4\|x\|\,\|\psi\|$, if $x \varepsilon X^+$, so that $\|Tx\| \le 4\|x\|$, for all positive x.

Repeating the argument above gives $\|T\| \le 16$, so that $(X^*)^+ \times X^+$ is a determining set, as desired.

Corollary 10. Let H be a Hilbert space, with innier product "$< >$". If the real and imaginary parts of $\{\frac{k!}{(k+n)!}\langle x, \ T^{k+n}x\rangle\}_{k=0}^{\infty}$ are positiue-definite, for all x in H, then T is C^n-scalar on $[-\|T\|, \ \|T\|]$.

Proof. We must show that $\{(x,x)|x$ is in $H\}\subseteq H\times H$ is a determing set (See definition 3).

First, Suppose T is symmetric, and $|<x,Tx>|\leq\|x\|^2$, for all x in H. A straightforward calculation gives $\langle x, Ty,\rangle=1/4\ [\langle x+y, T(x+y)\rangle-\langle x-y, T(x-y)\rangle+i(\langle x+iy, T(x+iy)\rangle-\langle x-iy, T(x-iy)\rangle]$

so that

$$|\langle x,Ty\rangle|\leq 1/4[\|x+y\|^2+\|x-y\|^2+\|x+iy\|^2+\|x-iy\|^2]=\|x\|^2+\|y\|^2.$$

Thus $\|T\|=\sup\limits_{\|x\|,\|y\|\leq 1}|\langle x,Ty\rangle|\leq 2.$

Now suppose T is an arbitrary bounded operator, such that $|\langle x,Tx\rangle|\leq\|x\|^2$, for all x in H. Let $R\equiv\frac{1}{2}(T+T^*)$, $S\equiv\frac{i}{2}(T^*-T)$. Then R and S are symmetric, and $T=R+iS$.

By what we've already shown, since $|\langle x,Rx\rangle|$ and $|\langle x,Sx\rangle|$ are $\leq\|x\|^2$, for all x, $\|R\|$ and $\|S\|$ are both≤ 2. Thus $\|T\|\leq 4$.

This concludes the proof.

One consequence is the well-known spectral theorem.

Corollary 11. A bounded symmetric operator on a Hilbert space is a spectrai operator of scalar type.

Proof. For any x,

$$\sum_{k,j}\alpha_k\bar{\alpha}_j\langle x,T^{k+j}x\rangle=\langle\sum\alpha_kT^kx,\sum\alpha_jT^jx\rangle=\|\sum\alpha_kT^kx\|^2\geq 0,$$

because T is symmetric; that is, $\{\langle x,T^kx\rangle\}_{k=0}^{\infty}$ is positive-definite. The result follows from Corollary 8.

Remark. Proposition 7 and Corollaries 8−11 can all be modified to conclude that the operators are C^n-scalar on $[0,\|T\|]$, by adding the hypothesis that $\sum_{k,j}\alpha_k\alpha_ja_{k+j+1}\geq 0$, for all finite sequences $\{\alpha_j\}$, where $a_k\equiv\frac{k!}{(k+n)!}\phi(T^{k+n}x)$. (See [8].

Example. For all future discussion, let

$$(2) \qquad (T_n f)(t) \equiv tf(t) + n(Jf)(t), \text{ where } (Jf)(t) \equiv \int_0^t f(s)ds.$$

Thebehavior of T_n becomes more transparent when one notes that $(T^k{}_n f)$ equals $(t^k J^n f)^{(n)}$, for all k, so that $g(T_n)$ may be defined by $g(T_n)f \equiv (gJ^n f)^{(n)}$, for any g with n continuous derivatives.

Theorem 12. Suppose Lebesgue measure is absolutely continuous with respect to v, with a bounded Radon-Nikodym derivative, $1 \leq p \leq \infty$, and v is finite. Then T_n, on $L^p([0,1],v)$, is C^n-scalar on $[0.1]$.

Proof. Let m be Lebesgue measure. We're given that $\|\frac{dm}{dv}\|_\infty$ and $v[0,1]$ are finiite. Note that this implies that $Jf(t) \equiv \int_0^t f(s)ds$ is a bounded operator on $L^p([0,1],v)$, by the following calculation (for $1 \leq p \leq \infty$).

$$\|Jf\|^p{}_p = \int_0^1 \left| \int_0^t f(s)ds \right|^p dv(t)$$

$$\leq \int_0^1 \int_0^1 |f(s)|^p ds \; dv(t), \text{ by Jensen's inequality,}$$

$$= \int_0^1 \int_0^1 |f(s)|^p dv(t) \frac{dm}{dv}(s) \; dv(s)$$

$$= \int_0^1 v[0,1] |f(s)|^p \frac{dm}{dv}(s) \; dv(s)$$

$$\leq v[0,1] \|\frac{dm}{dv}\|_\infty \|f\|_p{}^p.$$

Let $Kg(t) \equiv \int_t^1 g(s)ds$. For $p>1$, let q be such that $\frac{1}{q} + \frac{1}{p} = 1$; if $p=1$, let $q=\infty$. To apply Corollary 9, we calculate as follows, for any positive f in L^p, g in L^q.

$$\frac{k!}{(k+n!)} \int_0^1 g(T_n{}^{k+n} f)dv = \frac{k!}{(k+n)!} \int_0^1 g(\sum_{i=0}^n \binom{n}{i} (t^{k+n})^{(i)} J^i f)dv$$

$$= \int_0^1 g(\sum_{i=0}^n \binom{n}{i} \frac{k!}{(k+n-i)!} t^{k+n-i} J^i f)dv$$

$$= \sum_{i=0}^n \binom{n}{i} \int_0^1 t^k K^{n-i}(g(J^i f)dv)$$

(integrating by parts $(n-i)$ times in the ith term)

$$= \int_0^1 t^k (\sum_{i=0}^n \binom{n}{i} K^{n-i}(g(J^i f) dv)$$

Since $\left(\sum_{i=0}^{n}\binom{n}{i}K^{n-i}(g(J^i f)dv)\right.$ is positive, for all (f,g) in $(L^p)^+\times(L^q)^+$, a determining set, T_n is C^n-scalar on [0.1], by Corollary 9.

Corollary 13. Suppose g is bounded and measurable on [0,1] and bounded below by a positive number. Then T_n, on $L^p([0,1], g\ dt)$, is C^n-scalar on [0,1], when $1\leq p\leq\infty$.

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