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Representation of Bounded Operators by Hellingers Integral*

by

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> ***Abstract*** <

Bounded linear operators defined from a space with vector norm into (LF)-complete vector lattice are represented by Hellinger integration.

1. The Space $F(\tau, X)$

Suppose T is a non-empty set, τ an algebra of subsets of T. We shall denote with $\boldsymbol{A}(\tau, X)$ the set of additive bounded functions defined on τ with values in complete vector lattice X. Let μ be a real valued positive additive function defined on τ. A function F in $\boldsymbol{A}(\tau, X)$ is said to be μ-simple function if for all $A\varepsilon\tau$,

$$F(A)=\sum_{i=1}^{n} \mu(A\cap A_i)\ x_i \quad \cdots\cdots(1)$$

where $x_i\ \varepsilon X$ and $\{A_i\}$ $i=1,2,\ldots\ n$ is a τ-partition of T. $\boldsymbol{S}(\tau, X)$ denotes the family of μ-simple functions.

With order relation: $F_1 \leq F_2$ where $F_1(A) \leq F_2(A)$ for all $A\varepsilon\tau$, the space $\boldsymbol{A}(\tau, X)$ becomes a complete vector lattice, and $|||F|||=|F(T)|$ is a vector seminorm on this space. Define $\boldsymbol{F}(\tau, X)=\{F$: $F\varepsilon \boldsymbol{A}(\tau, X)$ and there exists a sequence $(F_n)_{n\varepsilon N}$, $F_n \varepsilon \boldsymbol{S}(\tau, X)$, such that $(\theta)\text{-}\lim|||F_n-F|||=0\}$.

Let Y be a complete vector lattice and m a positive additive function defined on τ with values in $(X, Y)^r$-the space of regular operators defined on X with values in Y. If $F\varepsilon \boldsymbol{S}(\tau, X)$, (given in form (1)), then

$$||F||=\sum_{i=1}^{n} m(A_i)\ (|X_i|)$$

is a vector seminorm on $\boldsymbol{S}(\tau, \mathrm{X})$. For functions in $\boldsymbol{S}(\tau, \mathrm{X})$ we define an integration of Hellinger Type:

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$$\int_T \frac{dmdF}{d\mu} = \sum_{i=1}^{n} m(A_i)\ (x_i) \quad \cdots\cdots(2)$$

where F is given in the form (1). It can be easily proved that

$$\left| \int_T \frac{dmdF}{d\mu} \right| \leq \|F\| \quad \cdots\cdots(3)$$

For more details, we refer to R. Cristescu (1) and F. Riesz and B.S. Nagy (3).

Remark (a) We can prove using Theorem 6.3 of Kawai (2) that in a (LF)-vector lattice with (θ)-continuous or regular topology, any sequence $(X_n)_{n\varepsilon N}$ of elements of X is (θ)-convergent, if

$$(\theta)\text{-}\lim_{k\to\infty} (X_k - X_{n_k}) = 0$$

for every subsequence$(x_{n_k})_{k\varepsilon N}$ of $(x_n)_{n\varepsilon N}$.

2. Bounded Operators on the Space $F(\tau, X)$

In what follows X will be a σ-regular complete vector lattice whereas Y will be a σ-regular (LF)-complete vector lattice.

Proposition 1. If a function m has property

$$m(A) \leq \mu(A)\ W, \quad \cdots\cdots(4)$$

for any $A\varepsilon\tau$ and $0 \leq W\varepsilon(\mathscr{X}, \mathscr{Y})^r$ -the space of (θ)-continuous regular operators, then integration (2) holds for any $F\varepsilon \boldsymbol{F}(\tau, X)$ and retains its property of linearity and positivity.

Proof. Let $F\varepsilon \boldsymbol{F}(\tau, X)$ then there exists a sequence $(F_n)_{n\varepsilon N}$, $F_n\ \varepsilon S(\tau, X)$ such that $(\theta)\text{-}\lim_{n\to\infty} |||F_n - F||| = 0$.

It follows that, $(\theta)\text{-}\lim_{k\to\infty} |||F_k - F_{n_k}||| = 0$ for anr any subsequence $(F_{n_k})_{k\varepsilon N}$ of $(F_n)_{n\varepsilon N}$. Using inequality (4), it implies $(\theta)\text{-}\lim_{k} \|F_k - F_{n_k}\| = 0$ for any subsequence $(F_{n_k})_{k\varepsilon N}$ of $(F_n)_{n\varepsilon N}$.

So, $\left| \sum_{i=1}^{n} \mu(A_i^k)\ (x_i^k) \right| \xrightarrow{\theta} 0$ in X implies $\sum_{i=1}^{n} m(A_i^k)(|x_i^k|) \xrightarrow{\theta} 0$ in Y.

Using inequality (3), we have

$$(\theta)\text{-}\lim_{k\to\infty} \left| \int_T \frac{dm\ d(F_k - F_{n_k})}{d\mu} \right| \leq (\theta)\text{-}\lim_{k\to\infty} \|F_k - F_{n_k}\| = 0.$$

Therefore,

$$(\theta)\text{-}\lim_{k\to\infty}\left|\int_T \frac{dm dF_k}{d\mu} - \int_T \frac{dm\ dF_{n_k}}{d\mu}\right| = 0.$$

Remark (a) of Section 1, implies that the sequence $\left(\int_T \frac{dm\ dF_n}{d\mu}\right)_{n\varepsilon N}$ is (θ)-convergent, since Y is σ-regular (LF)-complete vector lattice. Now put

$$(\theta)\text{-}\lim_{n\to\infty}\int_T \frac{dm\ dF_n}{d\mu} = \int_T \frac{dm\ dF}{d\mu}, \quad \cdots\cdots(5)$$

Inequalities (3) and (4) imply that, the limit in (5) is independet of the choice of the sequence $(F_n)_{n\varepsilon N}$. Obviously integral (5) is a positive linear operator defined on $\boldsymbol{F}(\tau, X)$.

It is noted that if F_X is a vector space with a vector seminorm (with values in X) and U a a linear operator, which maps F_X into complete vector lattice Y, for which there exists a positive operator $W\varepsilon(X,Y)_\theta{}^r$ such that

$$|U(F)| \leq (W|||F|||), \quad \cdots\cdots(6)$$

then U is bounded.

Proposition 2. Let m be a positive additive function on τ with values in the space $(X,Y)'$, satisfying the condition $m(A) \leq \mu(A)W$ for each $A\varepsilon\tau$ and $0 \leq W\varepsilon(X,Y)_\theta{}^r$, then the formula

$$U(F) = \int_T \frac{dm\ dF}{d\mu}, \quad \cdots\cdots(7)$$

defines a bounded positive linear operator on $\boldsymbol{F}(\tau, X)$ with values in Y.

Proof. First of all suppose that F is μ-simple function given by

$$F(A) = \sum_{i=1}^{n} \mu(A \cap A_i)\ x_i \text{ for any } A\varepsilon\tau.$$

Then

$$|U(F)| = \left|\int_T \frac{dm\ dF}{d\mu}\right| = \left|\sum_{i=1}^{n} m(A_i)\ (x_i)\right|$$

$$\leq \left|\sum_{i=1}^{n} \mu(A_i)\ W(x_i)\right| \leq W\left|\sum_{i=1}^{n} \mu(A_i)\ (x_i)\right| = W(|||F|||).$$

Thus

$$|U(F)| \leq W(|||F|||). \quad \cdots\cdots (8)$$

On the other hand, from proposition (1) it is clear that for any $F \varepsilon \boldsymbol{F}(\tau, X)$, (which is not μ-simple) there exists a sequence of μ-simple functions $(F_n)_{n \varepsilon N}$ such that

$$(\theta)\text{-}\lim_{n\to\infty} |||F_n - F||| = 0. \quad \cdots\cdots (9)$$

Therefore,

$$|U(F)| = \int_T \frac{dm\, dF}{d\mu} = (\theta)\text{-}\lim_{n\to\infty} \int_T \frac{dm\, dF_n}{d\mu},$$

and

$$|U(F)| \leq ((\theta)\text{-}\lim_{n\to\infty} W(|||F_n|||)).$$

Since W is (θ)-continuous, therefore,

$$|U(F)| \leq W((\theta)\text{-}\lim_{n\to\infty} |||F_n|||)$$

and

$$|U(F)| \leq W(|||F|||).$$

It follows that U is a bounded operator.

Since Hellinger Integral is positive and linear, thus U is a bounded, positive linear operator.

Proposition 3. If U is a bounded, positive linear operator(in sense of (6)) on $\boldsymbol{F}(\tau, X)$ with values in Y then there exists a positive additive function m defined on τ with values in $(X, Y)^r$, satisfying inequality (4) and

$$U(F) = \int_T \frac{dm\, dF}{d\mu} \text{ for all } F \varepsilon \boldsymbol{F}(\tau, X).$$

Proof. For every $A \varepsilon \tau$ and $x \varepsilon X$, we can define an operator F_A^x on τ with values in X, by $F_A^x(A_i) = \mu(A \cap A_i)\ (x)$ for all $A_i\ \varepsilon \tau$. Obviously, F_A^x is μ-simple.

Operator $m(A)$ defined on X with values in Y by $m(A)\ (x) = U(F_A^x)$ is a positive linear operator and m (as mapping: $A \longrightarrow m(A)$, of τ into $(X, Y)^r$) is positive, set additive and satisfies the inequality (4).

Case I. Suppose $F \varepsilon \boldsymbol{F}(\tau, X)$ is μ-simple (that is given by equality (1)), then

$$U(F(A))=U\left[\sum_{i=1}^{n}\mu(A\cap A_i)\ x_i\right]$$

$$=U\left[\sum_{i=1}^{n}F_{A_i}^{x_i}(A)\right]$$

$$=\sum_{i=1}^{n}U(F_{A_i}^{x_i}(A)).$$

Thus

$$U(F)=\sum_{i=1}^{n}U(F_{A_i}^{x_i})=\sum_{i=1}^{n}m(A_i)\ (x_i)=\int_T\frac{dm\ dF}{d\mu}$$

Case II. Suppose $F\varepsilon\boldsymbol{F}(\tau,X)$ is not μ-simple. Then there exists a sequence $(F_n)_{n\varepsilon N}$ of μ-simple functions such that

$$(\theta)\text{-}\lim_{n\to\infty}|||F_n-F|||=0$$

and

$$|U(F_n)-U(F)|=|U(F_n-F)|\leq W(|||F_n-F|||)\xrightarrow{\theta}0.$$

Hence

$$U(F)=(\theta)\text{-}\lim_n U(F_n) \quad\cdots\cdots\cdots(10)$$

From (10) and proposition (1), we arrive at

$$U(F)=\int_T\frac{dm\ dF}{d\mu} \quad\cdots\cdots\cdots(11)$$

Proposition (1), (2) and (3) imply the following theorem.

Theorem 4. If X is σ-regular complete vector lattice and Y σ-regular (LF)-complete vector lattice then general form of a bounded positive linear operator (in sense of (6)) defined on $\boldsymbol{F}(\tau,X)$ with values in Y is given by

$$U(F)=\int_T\frac{dm\ dF}{d\mu}$$

where m is a positive, additive function defined on τ with values in $(X,Y)^r$ satisfying inequality (4).

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