An Exact Triangle induced by a Short Exact Sequence which Splits.

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0. Introduction

Let $_RM$ be the category of R-modules and R-homomorphisms, where R is a commutative ing with unity. Also let COMP be the category of all chain complexes and chain transfornations. The differential modules in $_RM$ and some properties of them have been already introuced [4]. Also we obtained the exact triangle of homology modules of differential modules

5]. Using this facts, we proved that there exists the exact triangle of homology modules of
ifferential modules induced by chain complexes C, D and E, that is, a short exact sequence $C \longrightarrow D \longrightarrow E \longrightarrow 0$ in COMP. [Proposition 4].

The main object of this paper is to construct the exact triangle of direct sums of torsio. roducts induced by a short exact sequence $0 \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A'' \longrightarrow 0$ in $A' \longrightarrow 0$ in A

Most of notations are taken from (2).

1. Preliminaries

Definition 1. Let X be an R-module and (C, ∂) a chain complex:

$$C: \cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

Then C is called projective resolution of X if

- $(1) C_{-1} = X$
- (2) $C_n=0$ for every n<-1
- (3) C_n is a projective R-module for every $n \ge 0$.

Let X and Y be arbitrarily given R-modules. Select any projective resolution C of the module X

$$C: \cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

and consider its tensor product $C \otimes Y$ which is the sequence

$$C \otimes Y : \cdots \longrightarrow C_{n+1} \otimes Y \xrightarrow{\partial *} C_n \otimes Y \xrightarrow{\partial *} C_{n-1} \otimes Y \longrightarrow \cdots$$

where $\partial *$ stands for the tensor product $\partial \otimes i$ of the homomorphism ∂ and the identity endomorphism i of the module Y. Since $\partial * \circ \partial * = (\partial \otimes i) \circ (\partial \otimes i) = (\partial \circ \partial) \otimes (i \circ i) = 0 \otimes i = 0$, $C \otimes Y$ is a semi-exact sequence and so $C \otimes Y$ is a chain complex. Thus for every integer n, the n-dim-

ensional homology module $H_n(C \otimes Y)$ of $C \otimes Y$ is defined.

Proposition 2. For any two projective resolution C, D of the module X,

$$C: \cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

$$D: \cdots \longrightarrow D_{n+1} \xrightarrow{\partial} D_n \xrightarrow{\partial} D_{n-1} \longrightarrow \cdots$$

we have $H_n(C \otimes Y) \approx H_n(D \otimes Y)$ for every integer n.

Proof. The proof may be found in [2, p. 131].

The above module $H_n(C \otimes Y)$ depends essentially only on the integer n and given R-modules X, Y. Thus we make the following definition.

Definition 3. For every integer n, the R-module $H_n(C \otimes Y)$ is said to be the n-dimensional torsion product of the given modules X, Y and is denoted by the symbol $Tor_n^R(X, Y)$.

Proposition 4. Let

$$0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

be a given short exact sequence in COMP. Then there exists an exact triangle of homology modules of differential modules induced by chain complexes C, D and E:

$$H(X) \longrightarrow H(Y)$$
 $H(Z)$

where
$$X = \bigoplus_{n \in \mathbb{Z}} C_n$$
, $Y = \bigoplus_{n \in \mathbb{Z}} D_n$ and $Z = \bigoplus_{n \in \mathbb{Z}} E_n$.

Proof. It follows from [3].

Proposition 5. If a short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$
 in _RM

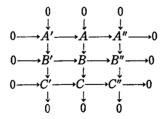
splits, then for any R-module K, the sequence

$$0 \longrightarrow A' \otimes K \longrightarrow A \otimes K \longrightarrow A'' \otimes K \longrightarrow 0$$
 in M

is exact.

Proof. It follows from (2, p. 66).

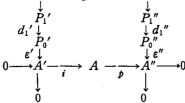
Proposition 6. (3×3 lemma) Consider the commutative diagram in _RM:



If the columns are exact and if the bottom two rows are exact, then the top row is exact.

Proof. It follows from (1, p. 175).

Proposition 7. Consider the diagram in _RM:



where the columns are projective resolutions and the row is exact. Then there exists a protive resolution of A and chain transformations so that the columns form an exact sequence COMP.

Proof. By induction, it sufficies to complete the 3×3 diagram

$$\begin{array}{cccc}
0 & 0 & & & \\
K_0' & & K_0'' & & \\
P_0' & & P_0'' & & \\
0 & A' & A & \rightarrow A'' & 0
\end{array}$$

where the rows and columns are exact and P_0' , P_0'' are projective. Define $P_0 = P_0' \oplus P_0''$, $P_0' \to P_0$ by $x' \to (x', 0)$, and $p_0 : P_0 \to P_0''$ by $(x', x'') \to x''$.

$$P_{0}' \xrightarrow{i_{\bullet}} P_{0} \xrightarrow{p_{\bullet}} P_{0}''$$

$$\downarrow \varepsilon' \quad \varepsilon \downarrow \qquad \downarrow \varepsilon''$$

$$A' \xrightarrow{i_{\bullet}} A \xrightarrow{i_{\bullet}} A'' \xrightarrow{i_{\bullet}} 0$$

is clear that P_0 is projective and that

$$0 \longrightarrow P_0' \xrightarrow{i_o} P_0 \xrightarrow{p_o} P_0'' \longrightarrow 0$$

exact. Since P_0'' is projective, there is a homomorphism $\sigma: P_0'' \longrightarrow A$ with $p \circ a = \varepsilon''$ Define $P_0 \longrightarrow A$ by $\varepsilon: (x', x'') \longrightarrow (i \circ \varepsilon') (x') + \sigma(x'')$. It is an verification that, if $K_0 = \ker \varepsilon$, the sulting 3×3 diagram commutes. Exactness of the top is the 3×3 Lemma.

$$0 \longrightarrow \overset{0}{K_0}' \longrightarrow \overset{0}{K_0} = \ker \varepsilon \longrightarrow \overset{0}{K_0}'' \longrightarrow 0$$

$$0 \longrightarrow \overset{0}{P_0}' \longrightarrow \overset{1}{P_0} = P_0' \oplus P_0'' \longrightarrow 0$$

$$\varepsilon' \downarrow \qquad \qquad \downarrow \varepsilon \qquad \qquad \downarrow \varepsilon''$$

$$0 \longrightarrow A' \longrightarrow \overset{1}{i} \longrightarrow \overset{1}{A} \longrightarrow \overset{1}{p} \longrightarrow \overset{1}{A}'' \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

2. Main theorem

Theorem 8, Let

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$
 in _RM

2 a given short exact sequence which splits. Then there exists an exact triangle of direct sums f torsion products, that is,

$$\bigoplus_{n \in \mathbb{Z}} Tor_{n}^{R}(A', K) \longrightarrow \bigoplus_{n \in \mathbb{Z}} Tor_{n}^{R}(A, K)$$

$$\bigoplus_{n \in \mathbb{Z}} Tor_{n}^{R}(A'', K)$$

Proof. By Proposition 7, there exists an exact sequence

$$0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$
 in $COMP$,

where C, D and E are projective resolutions A', A and A'' respectively. This means that, for

every integer $n \in \mathbb{Z}$,

$$0 \longrightarrow C_n \longrightarrow D_n \longrightarrow E_n \longrightarrow 0$$

is exact in $_RM$.

By Proposition 5

$$0 \longrightarrow C_n \otimes K \longrightarrow D_n \otimes K \longrightarrow E_n \otimes K \longrightarrow 0$$

is exact in _RM. Thus

$$0 \longrightarrow C \otimes K \longrightarrow D_n \otimes K \longrightarrow E \otimes K \longrightarrow 0$$

is exact in COMP. By Proposition 4, there exists an exact triangle of homology modules of differential modules induced by chain complexes $C \otimes K$, $D \otimes K$ and $E \otimes K$:

$$\bigoplus_{n\in\mathbb{Z}}(C_n\otimes K))\longrightarrow H(\bigoplus_{n\in\mathbb{Z}}(D_n\otimes K))$$

$$\downarrow H(\bigoplus_{n\in\mathbb{Z}}(E_n\otimes K))$$

Also we obtain $H(\bigoplus_{n\in Z}(C_n\otimes K))=\bigoplus_{n\in Z}H_n(C\otimes K)=\bigoplus_{n\in Z}Tor_n^R(A',K)$. Similarly $H(\bigoplus_{n\in Z}(D_n\otimes K))=\bigoplus_{n\in Z}Tor_n^R(A,K)$ and $H(\bigoplus_{n\in Z}(E_n\otimes K))=\bigoplus_{n\in Z}Tor_n^R(A'',K)$. This proof is complet.

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