國數學教育學會誌<數學教育>
35. 12, Vol. XXIV, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1985, Vol. XXIV, No. 1.

Nonlinear Ergodic Theorems for Asymptotically Nonexpansive Mapping in Hilbert Space

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l. Introduction

We are interested in proving the existence of limits of time averages. The recent develo-
ıents in the ergodic theory of nonlinear mappings in Hilbert sapce started with the results
Baillon([1]). Baillon considered a nonexpansive mapping T of a real Hilbert space H into
elf. He proved that if T has fixed points in H then for all x in H, the Cesàro means:

$$S_n x=\frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

ıverges strongly as $n\longrightarrow\infty$ to a fixed point of T([1]). A corresponding theorem for a strongly
ıtinuous one parameter semigroup of nonexpansive mapping $S(t)$, $t\geqq 0$, was given soon after
illon's work by Baillon and Brezis ([2]). Similar results were also obtained by N. Hirano
d W. Takahashi ([5]) for an asymptotically nonexpansive mapping. But above results are
the cases for existence of weak limit. From example of A. Genel and J. Lindenstrauss
3]), it follows that there exists a nonexpansive mapping such that the Cesàro mean don't
ıverge strongly. A. Pazy([4]) therefore gave some further assumptions on the mapping in
ler to assure the strong convergence of the Cesàro mean. J.K. Kim and K.S. Ha([6])
rived the same results of A. Pazy by reducing the continuous parameter case to the discrete
rameter case. The purpose of this paper is to give the proofs of the strong convergence of
Pazy's results for an asymptotically nonexpansive mapping.

2. Main Theorems

Let C be a closd convex subset of a real Hilbert space H, and T be a mapping of C into
elf. T is said to be asymptotically nonexpansive if for all $x, y\in C$,

$$\|T^i x-T^i y\|\leqq(1+\alpha_i)\|x+y\| \text{ for } i=1,2,\cdots,$$

ıere $\lim_{i\to\infty}\alpha_i=0$.

Let we define the Cesàro means:

$$S_n x=\frac{1}{n}\sum_{k=0}^{n-1} T^k x \quad \text{for all } x \text{ in } C.$$

Let $F(T)=\{x\mid Tx=x\}$ be the set of all fixed points of T in C for every x in C.
The following Theorem 1 is well-known ([5]).

Theorem 1. *Let C be a closed convex subset of a real Hilbert space H, and T be an asympto-*

tically nonexpansive self-mapping on C such that for each x in C, $\{T^n x\}$ is bounded then $\{S_n x\}$ converges weakly to a fixed point of T for all x in C.

Lemma 2. *Let C and T satisfy the same assumptions as in Theorem 1. Then for every $x \in C$ and $\varepsilon > 0$, there exists $k_0 > 0$ such that for each $m \geqq k_0$, there is $N_m > 0$ satisfying*

$$\|S_n x - T^m S_n x\| < \varepsilon \quad \text{for all } n \geqq N_m.$$

Proof. For all v in H,

$$\begin{aligned}\|S_n x - v\|^2 &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} (T^k x - v) \right\|^2 \\ &= \frac{1}{n^2} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} (T^k x - v,\ T^j x - v) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \| T^k x - v\|^2 - \frac{1}{n} \sum_{i=0}^{n-1} \| T^k x - S_n x\|^2.\end{aligned}$$

Put $v = T^k S_n x$ for $k \leqq n$, then

$$\begin{aligned}\|S_n x - T^k S_n x\|^2 &= \frac{1}{n} \sum_{i=0}^{k-1} \|T^i x - T^k S_n x\|^2 + \frac{1}{n} \sum_{i=k}^{n-1} \|T^i x - T^k S_n x\|^2 \\ &\qquad - \frac{1}{n} \sum_{i=0}^{n-1} \|T^i x - S_n x\|^2 \\ &\leqq \frac{1}{n} \sum_{i=0}^{k-1} \|T^i x - T^k S_n x\|^2 + (1 + \alpha_k)^2 \frac{1}{n} \sum_{i=0}^{n-k-1} \|T^i x - S_n x\|^2 \\ &\qquad - \frac{1}{n} \sum_{i=0}^{n-1} \|T^i x - S_n x\|^2 \\ &\leqq \frac{1}{n} \sum_{i=0}^{k-1} \|T^i x - T^k S_n x\|^2 + (2\alpha_k + \alpha_k^2) \frac{1}{n} \sum_{i=0}^{n-k-1} \|T^i x - S_n x\|^2.\end{aligned}$$

Let d be the diameter of $\{T^n x : n = 1, 2, \cdots\}$, then, for all $i, n \in N$, $\|T^i x - S_n x\|^2 \leqq d^2$. For $\varepsilon > 0$, sthere exists $K_0 > 0$ such that

$$(2\alpha_k + \alpha_k^2) < \frac{\varepsilon^2}{2d^2} \quad \text{for all } k \geqq K_0.$$

Thus if $k \geqq K_0$, there $(2\alpha_k + \alpha_k^2) \frac{1}{n} \sum_{i=0}^{n-k-1} \|T^i x - S_n x\|^2 < \frac{\varepsilon^2}{2}$, and for $m \geqq K_0$, there exists $N_m > ($ uch that

$$\frac{1}{n} \sum_{i=0}^{k-1} \|T^i x - T^m S_n x\|^2 < \frac{\varepsilon^2}{2} \quad \text{for all } n \geqq N_m.$$

Hence, we have

$$\|S_n x - T^m S_n x\|^2 < \varepsilon^2.$$

Let B be a unit ball of l^2. It is shown that there exists a nonexpansive mapping T and ε point x in B such that $\{S_n x\}$ does not converge strongly in l^2 ([3]). Hence we will give som further assumptions on the mapping in order to assure the strong convergence of the time av erages, and we will prove the following two theorems, using above Theorem 1 and Lemma 2

Theorem 3. *Let C be a closed convex subset of a real Hilbert space H, and T be an asym ptotically nonexpansive self-mapping on C. If T is a compact mapping on C and $\{T^n x\}$ is bou nded for each x in C, then F(T) is nonempty, and for all $x \in C$, $\{S_n x\}$ converges strongly t a fixed point of T.*

Proof. It follows from ([7], [8]) that $F(T)$ is nonempty. Let $\{S_{n_k} x\}$ be an arbitrary subse quence of $\{S_n x\}$ then $\{S_{n_k} x\}$ is bounded. Since T is compact, T^m is compact, hence there exist a subsequence $\{S_{n_{k_j}} x\}$ of $\{S_{n_k} x\}$ such that $\{T^m S_{n_{k_j}} x\}$ converges strongly to a point $u \in C$.

By the way, since $\{S_n x\}$ converges weakly to a fixed point of T by Theorem 1, also by emma 2,

$$0\leqq\|p-u\|\leqq\varliminf_{j\to\infty}\|S_{n_{k_j}}x-T^m S_{n_{k_j}}x\|\longrightarrow 0,$$

ence $p=u$. Therefore, we have

$$\|S_{n_{k_j}}x-p\|\leqq\|S_{n_{k_j}}x-T^m S_{n_{k_j}}x\|+\|T^m S_{n_{k_j}}x-p\|.$$

implies that $\{S_{n_{k_j}}x\}$ converges strongly to p in $F(T)$ as $n_{k_j}\longrightarrow\infty$. Hence $\{S_n x\}$ converges :rongly to a point $p\in F(T)$ for all $x\in C$.

Theorem 4. *Let C be a closed convex subset of a real Hilbert pace H, and T be an asymptot-:ally nonexpansive self-mapping on C. If $I-T^m$ maps a bounded closed set of a closed set and $T^n x\}$ is bounded for all x in C, then $\{S_n x\}$ converges strongly to a fixed point of T.*

Proof. In order to prove the theorem, we must show that every subsequence $\{S_{n_k}x\}$ of $\{S_n x\}$ as a strongly convergent subsquence to a fixed point of T. By Therem 1, for every x in C, $S_n x\}$ converges weakly to a point $p\in F(T)$, hence $\{S_n x\}$ converges weakly to a point $\in F(T^m)$.

Let $G=\{\overline{S_{n_k}x : k=1,2,3,\cdots}\}$ (strong closure of $\{S_{n_k}x\}$), then G is a closed and bounded set. Ience $(I-T^m)G$ is a closed set by assumption.

On the other hand, by Lemma 2, since $\{S_{n_k}x\}\subset G$, for such $m>0$

$$0\in\overline{(I-T^m)G}=(I-T^m)G.$$

'herefore, there is an element $u\in G$, such that $(I-T^m)u=0$. Since G is strong closure of $\{S_{n_k}x\}$, here exists a subsequence $\{S_{n_{k_j}}x\}$ of $\{S_{n_k}x\}$ such that $\{S_{n_{k_j}}x\}$ converges to a pcint $u\in G$. By the vay, since p is weak limit of $\{S_n x\}$, u is necessary equal to p by uniqueness of weak limit.

Consequently $\{S_{n_{k_j}}x\}$ converges strongly to a fixed point p of T. Hence $\{S_n x\}$ converges stro-ıgly to a point $p\in F(T)$.

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